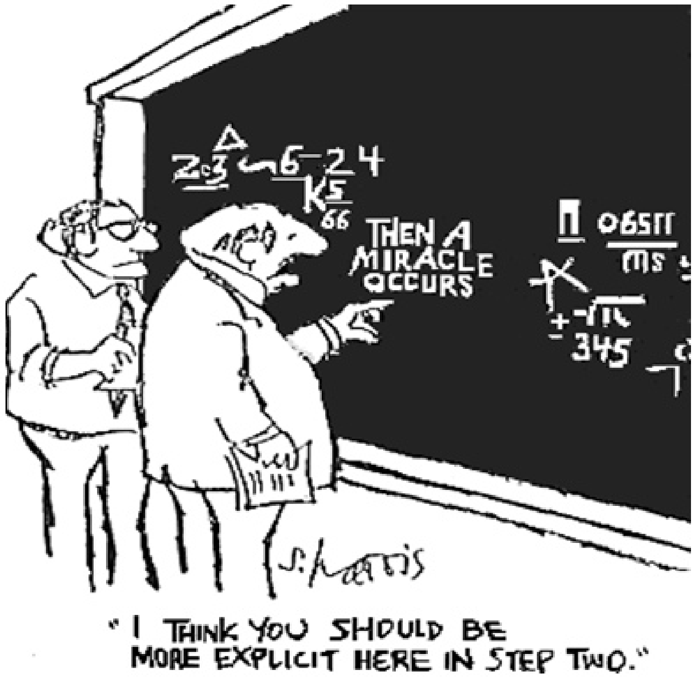
CSE 311: Foundations of Computing

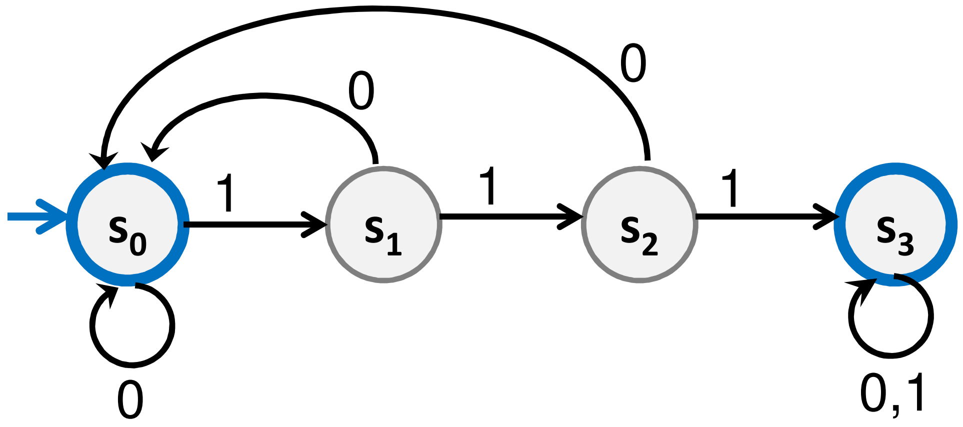
Lecture 24: NFAs and their relation to REs & DFAs



Recall: DFAs

- States
- Transitions on input symbols
- Start state and final states
- The “language recognized” by the machine is the set of strings that reach a final state from the start

Old State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3

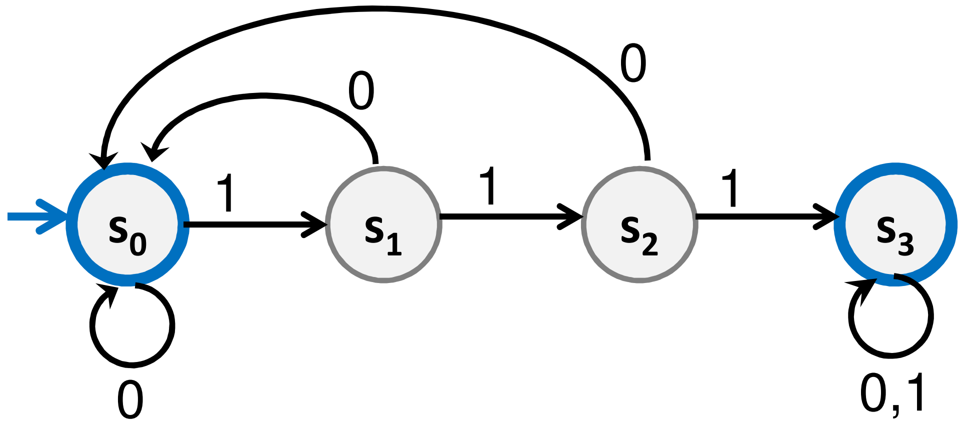

$$\begin{tikzcd}
\arrow[r] & s_0 \arrow[loop below, "0"] \arrow[r, "1"] & s_1 \arrow[l, bend right, "0"'] \arrow[r, "1"] & s_2 \arrow[ll, bend right=50, "0"'] \arrow[r, "1"] & s_3 \arrow[loop below, "{0,1}"]
\end{tikzcd}$$

Recall: DFAs

- Each machine designed for strings over some fixed alphabet Σ.

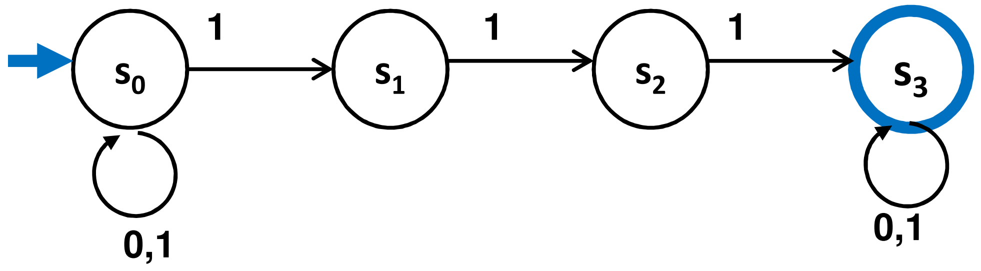
- Must have a transition defined from each state for *every* symbol in Σ.

Old State	0	1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_3
s_3	s_3	s_3



Last Time: Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
  - Also can have edges labeled by empty string ε
- Definition: x is in the language recognized by an NFA if and only if <u>some</u> valid execution of the machine gets to an accept state



Three ways of thinking about NFAs

- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)

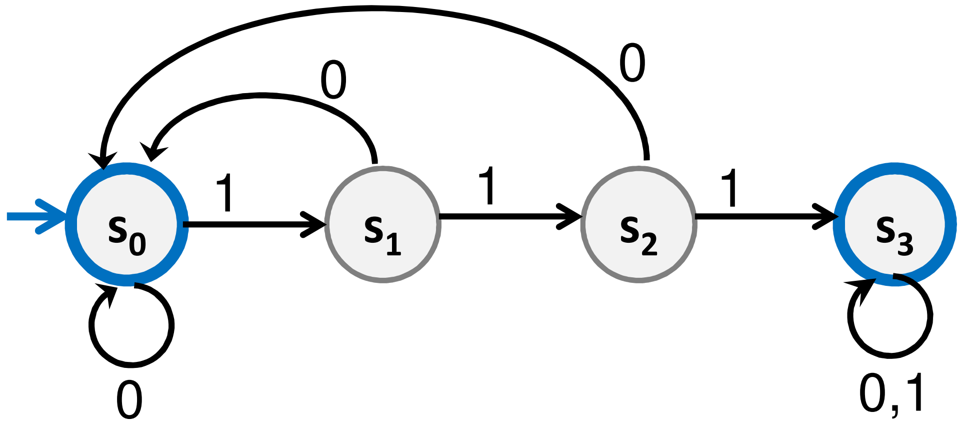
- Outside observer: Is there a path labeled by x from the start state to some accepting state?

- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

Path Labels

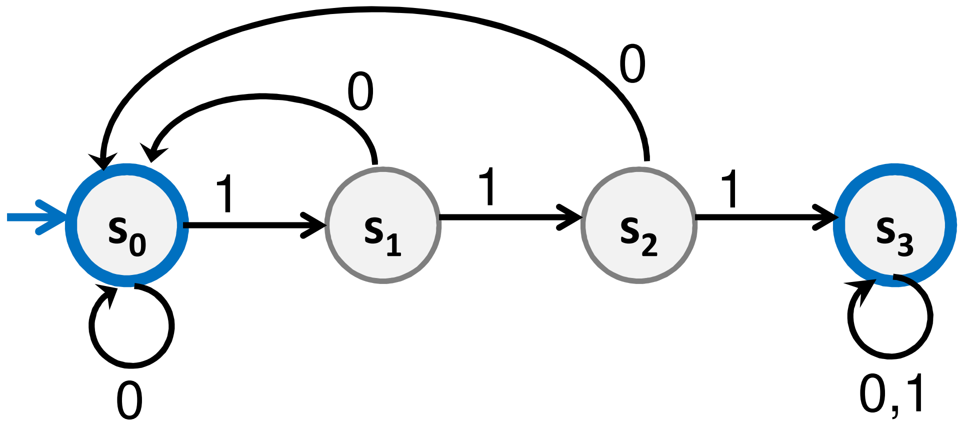
Def: The label of path $v_0, v_1, \ldots, v_n$ is the <u>concatenation</u> of the labels of the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$

Example: The label of path s_0, s_1, s_2, s_0, s_0 is 1100



Deterministic Finite Automata (DFA)

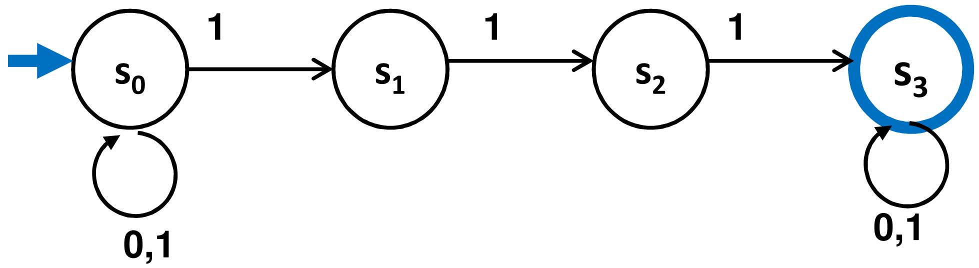
- Def: x is in the language recognized by an DFA if and only if x labels a path from the start state to some final state



- Path $v_0, v_1, \ldots, v_n$ with $v_0 = s_0$ and label x describes a correct simulation of the DFA on input x
  - i-th step must match the i-th character of x (there may be options for which label to take between vertices).

Nondeterministic Finite Automata (NFA)

- Graph with start state, final states, edges labeled by symbols (like DFA) but
  - Not required to have exactly 1 edge out of each state labeled by each symbol— can have 0 or >1
  - Also can have edges labeled by empty string ε
- **Definition:** x is in the language recognized by an NFA if and only if x labels <u>some</u> path from the start state to an accepting state



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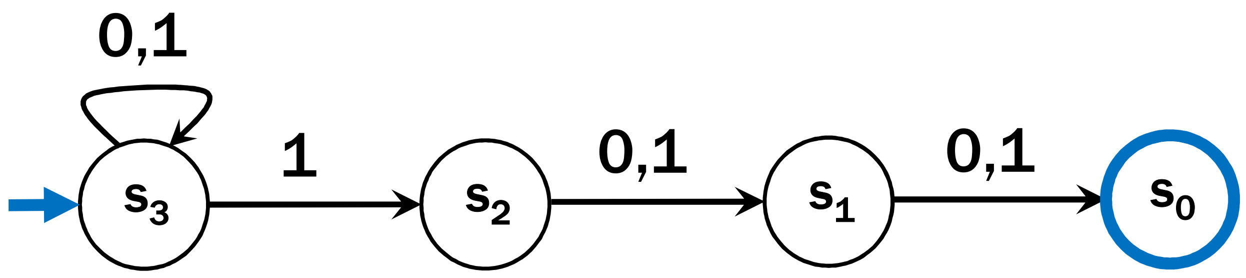
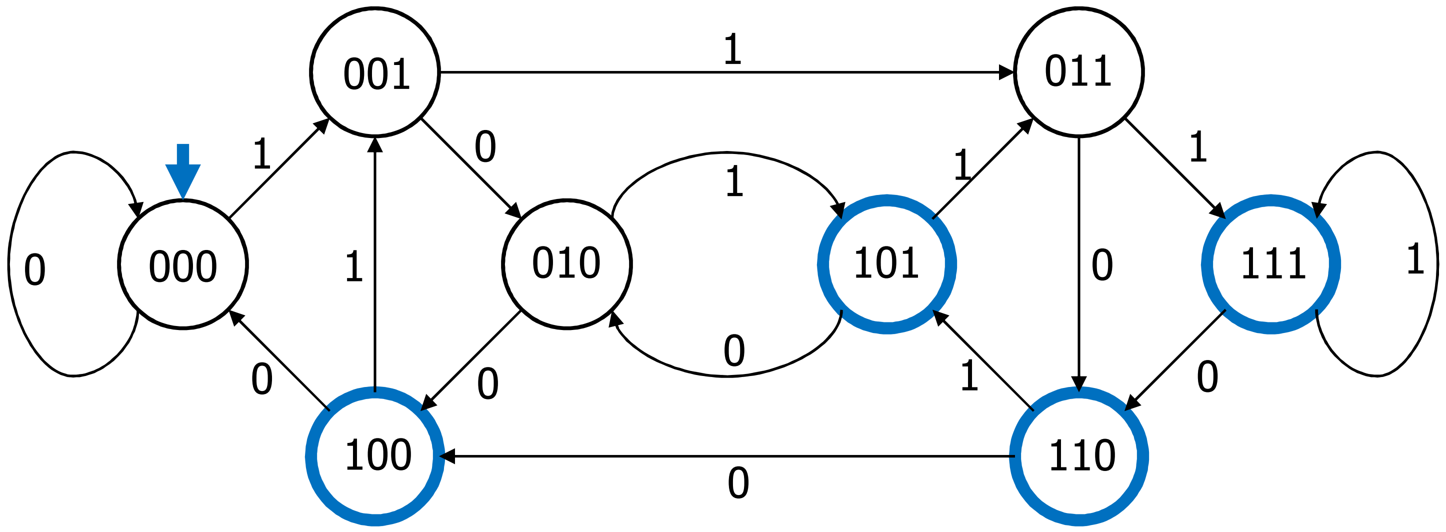
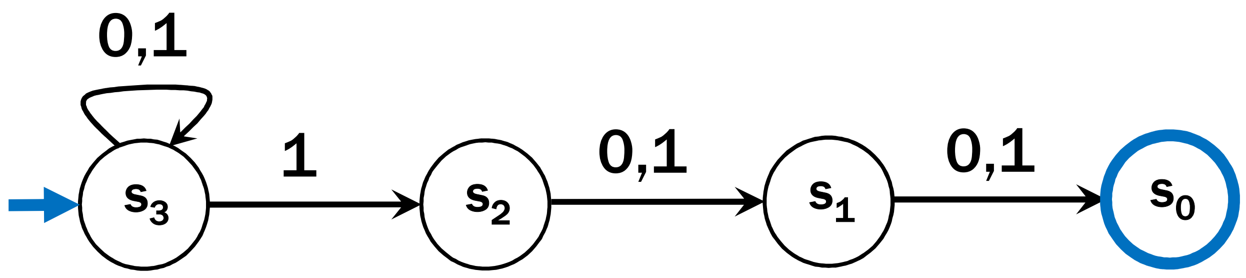
- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

Compare with the smallest DFA

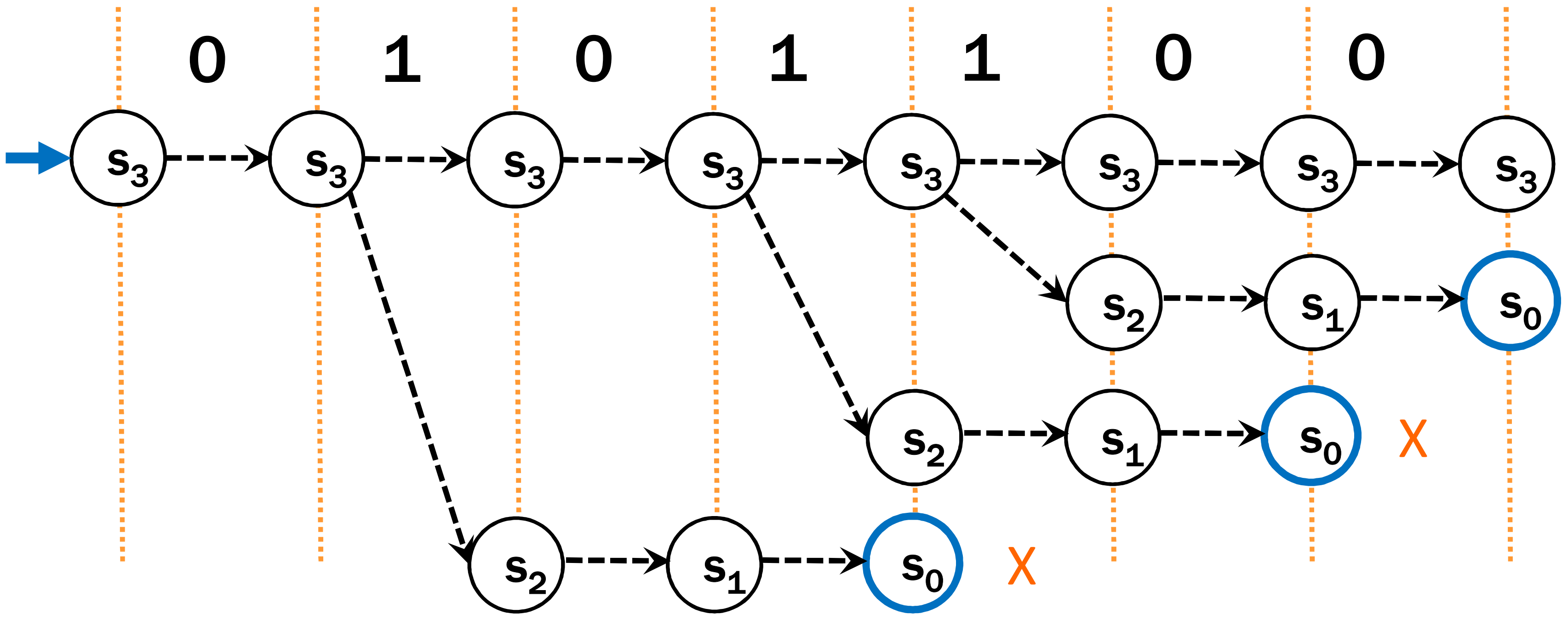




Parallel Exploration view of an NFA



Input string 0101100



Summary of NFAs

- Generalization of DFAs
  - drop two restrictions of DFAs
  - every DFA <u>is</u> an NFA
- *Seem* to be more powerful
  - designing is easier than with DFAs
- *Seem* related to regular expressions

The story so far...

$$\boxed{\text{REs}} \subseteq \boxed{\text{CFGs}}$$

$$\boxed{\text{DFAs}} \subseteq \boxed{\text{NFAs}}$$

NFAs and regular expressions

Theorem: For any set of strings (language) A, if there is a regular expression for A then there is an NFA that recognizes A.

Proof idea: Structural induction based on the recursive definition of regular expressions…

Regular Expressions over Σ

- **Basis:**
  - ε is a regular expression
  - *a* is a regular expression for any $a \in \Sigma$
- **Recursive step:**
  - If A and B are regular expressions then so are:

    A $\cup$ B

    AB

    A*

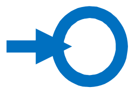
Base Case

- Case ε:

- Case *a*:

Base Case

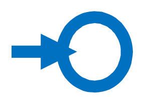
- Case ε:



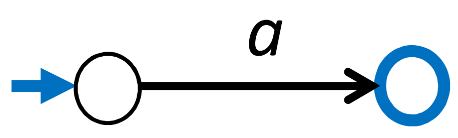
- Case *a*:

Base Case

- Case ε:

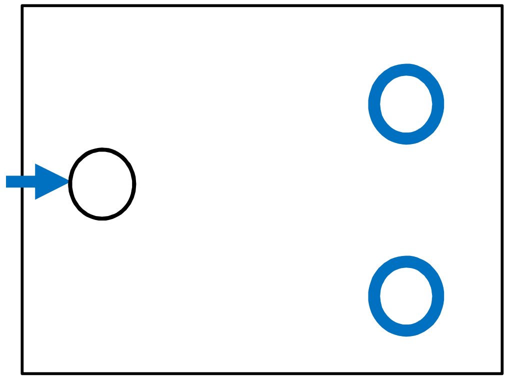


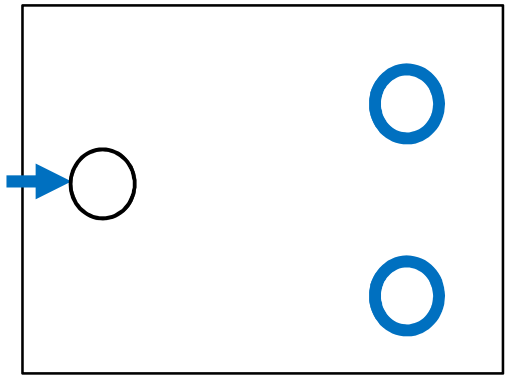
- Case *a*:



Inductive Hypothesis

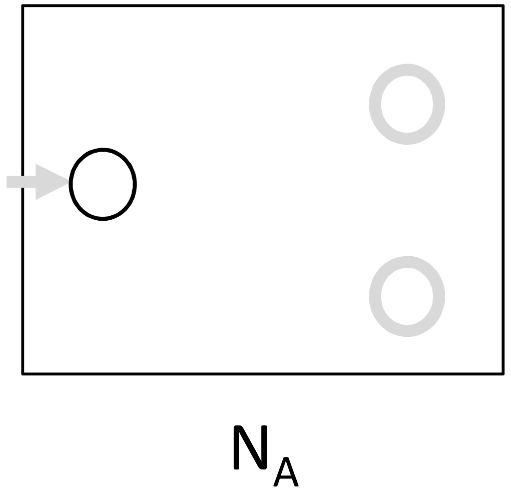
- Suppose that for some regular expressions A and B there exist NFAs N_A and N_B such that N_A recognizes the language given by A and N_B recognizes the language given by B

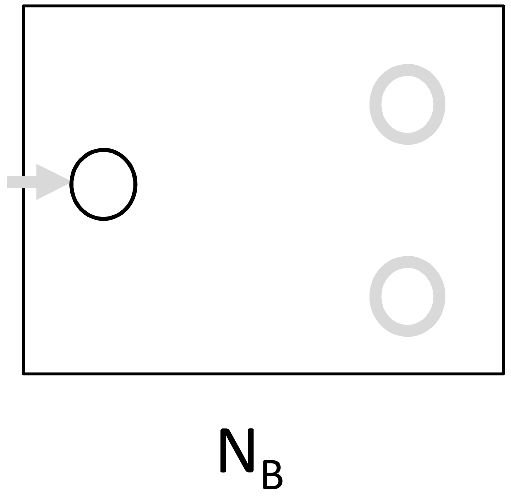

N_A


N_B

Inductive Step

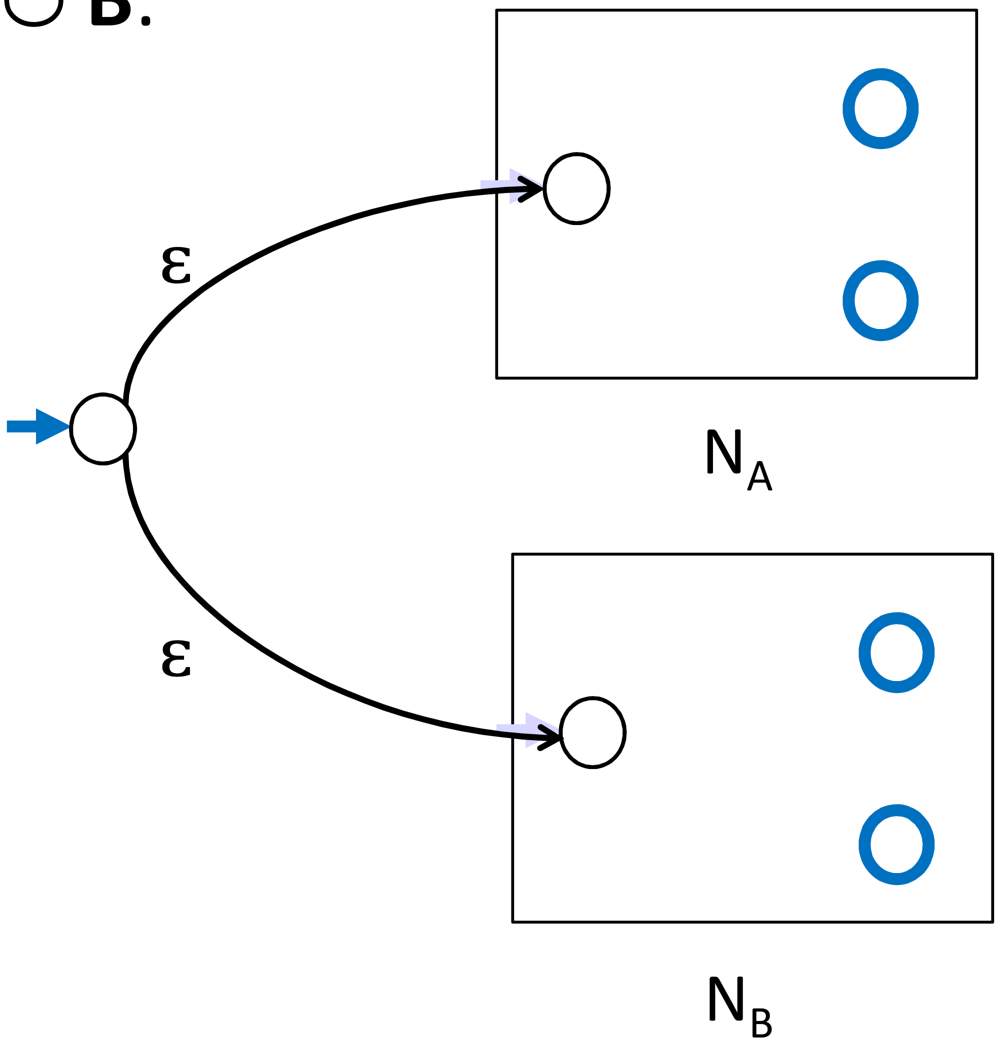
Case $A \cup B$:





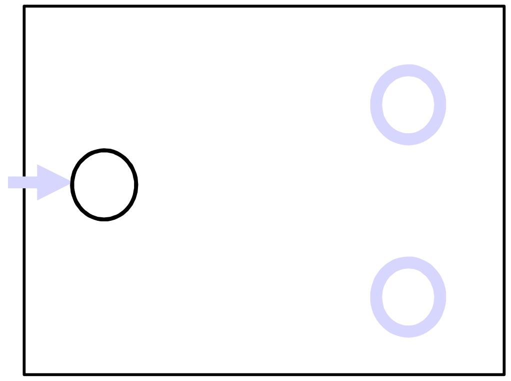
Inductive Step

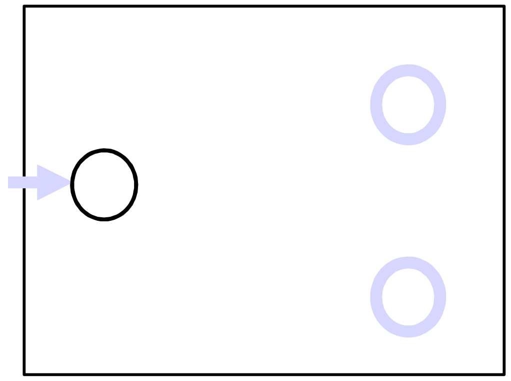
Case A ∪ B:



Inductive Step

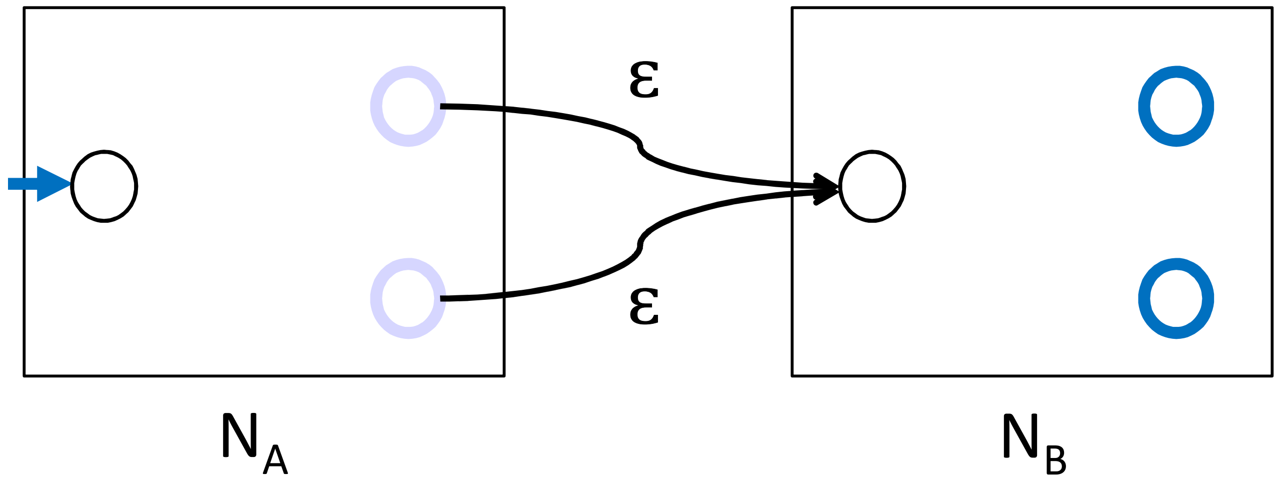
Case AB:


N_A


N_B

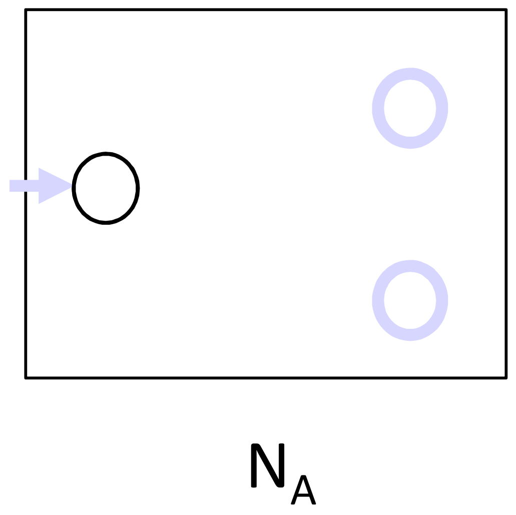
Inductive Step

Case AB:



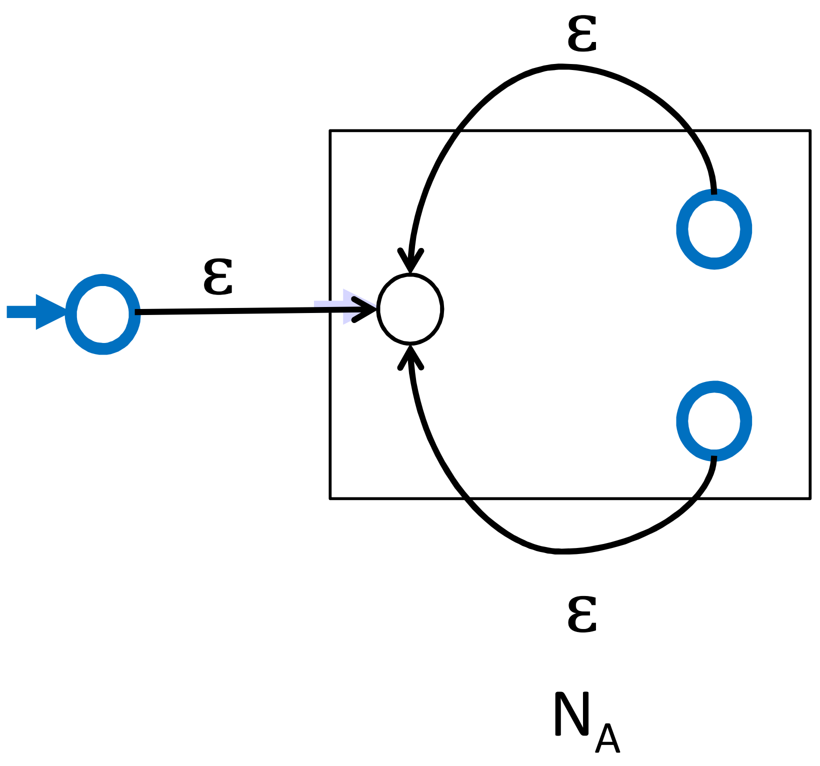
Inductive Step

Case A*



Inductive Step

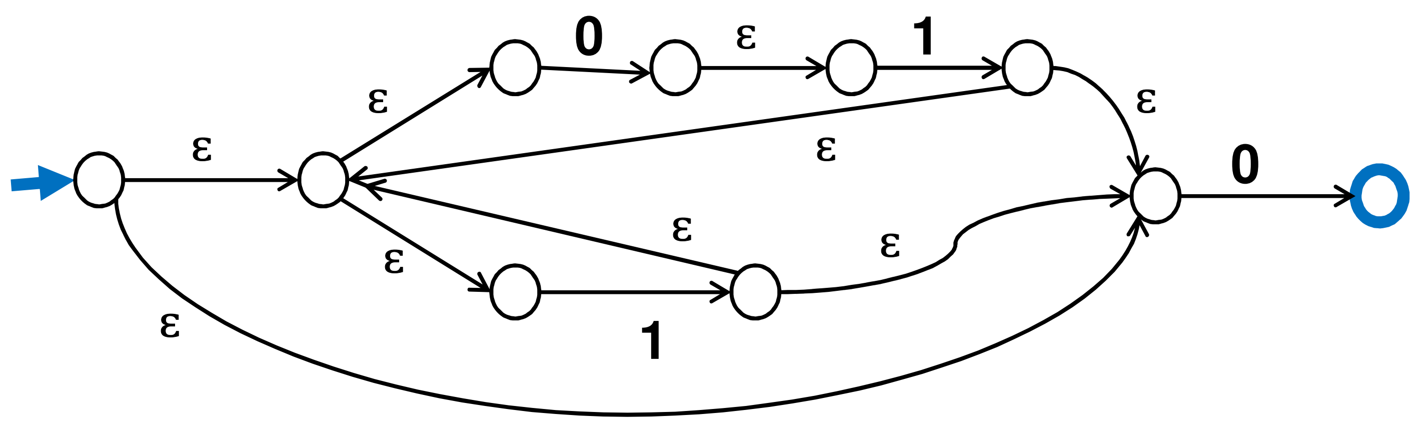
Case A*



Build an NFA for $(01 \cup 1)^*0$

Solution

(01 ∪1)*0



The story so far...

$$\begin{tikzcd}
\boxed{\text{REs}} \arrow[r, phantom, "\subseteq"] & \boxed{\text{CFGs}} \\
\boxed{\text{DFAs}} \arrow[r, phantom, "\subseteq"] & \boxed{\text{NFAs}} \arrow[ul, phantom, "\subseteq" sloped]
\end{tikzcd}$$

NFAs and DFAs

Every DFA is an NFA

- DFAs have requirements that NFAs don’t have

Can NFAs recognize more languages?

NFAs and DFAs

Every DFA is an NFA

– DFAs have requirements that NFAs don't have

Can NFAs recognize more languages? No!

Theorem: For every NFA there is a DFA that recognizes exactly the same language

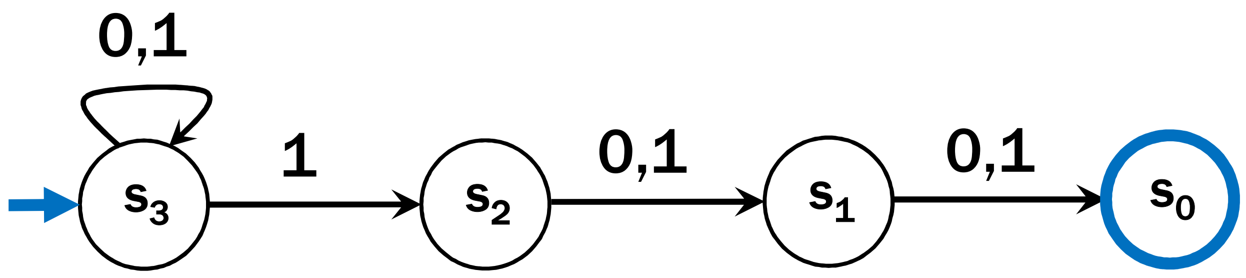
Three ways of thinking about NFAs

- Outside observer: Is there a path labeled by x from the start state to some final state?

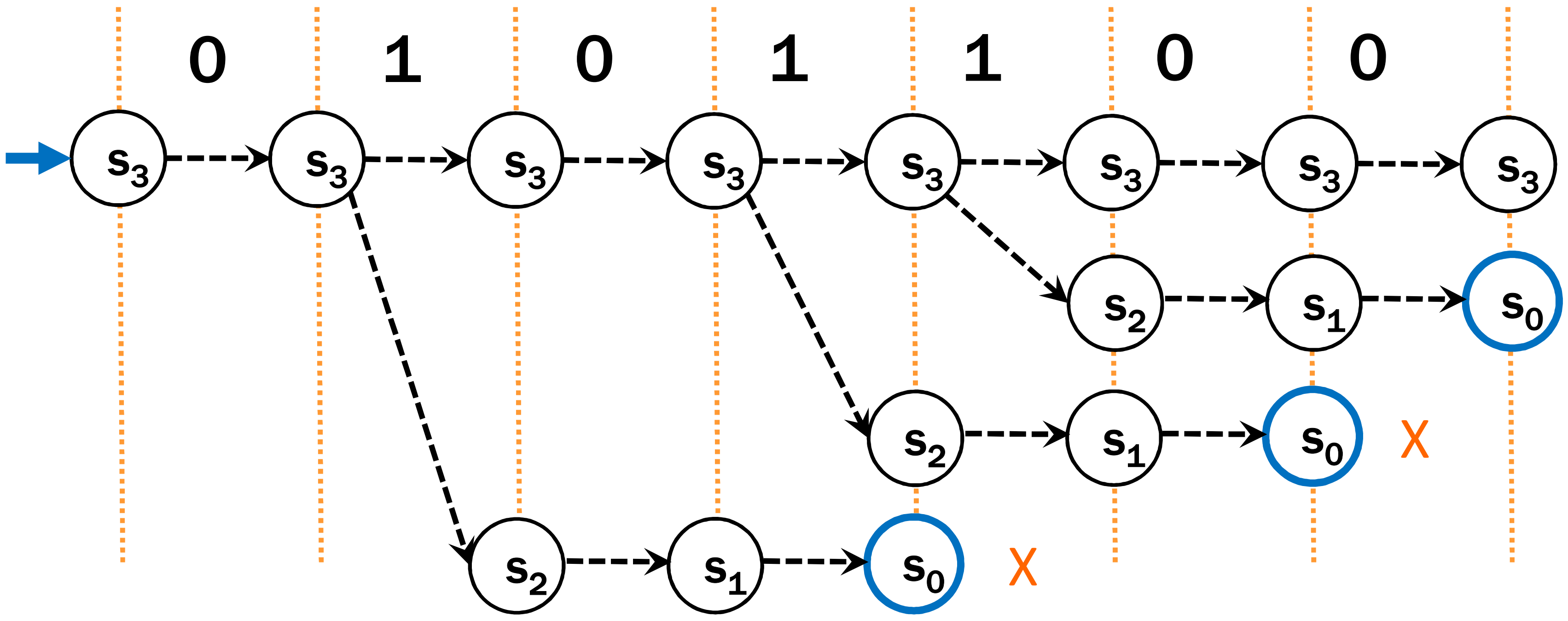
- Perfect guesser: The NFA has input x and whenever there is a choice of what to do it magically guesses a good one (if one exists)

- Parallel exploration: The NFA computation runs all possible computations on x step-by-step at the same time in parallel

Parallel Exploration view of an NFA



Input string 0101100



Conversion of NFAs to a DFAs

- Construction Idea:
  - The DFA keeps track of ALL states reachable in the NFA along a path labeled by the input so far

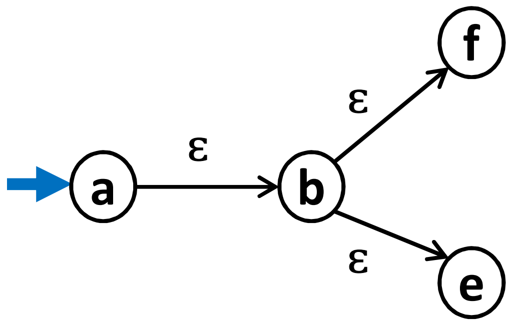
    (Note: not all *paths*; all *last states* on those paths.)

  - There will be one state in the DFA for each *subset* of states of the NFA that can be reached by some string

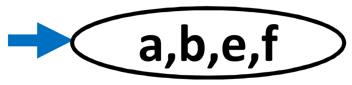
Conversion of NFAs to a DFAs

New start state for DFA

– The set of all states reachable from the start state of the NFA using only edges labeled ε



NFA

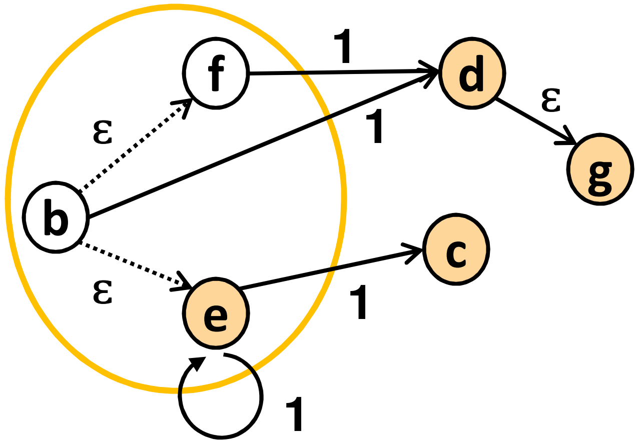


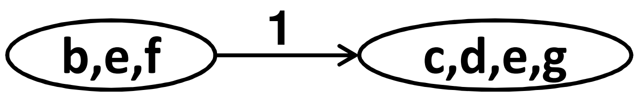
DFA

Conversion of NFAs to a DFAs

For each state of the DFA corresponding to a set S of states of the NFA and each symbol a

- **Add an edge labeled** a **to state corresponding to** T, **the set of states of the NFA reached by**
  - starting from some state in S, then
  - following one edge labeled by a, and
    then following some number of edges labeled by ε
- T **will be** ∅ **if no edges from** S **labeled** a **exist**

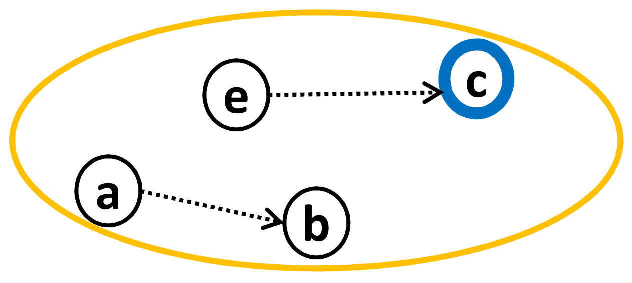




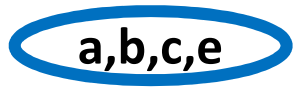
Conversion of NFAs to a DFAs

Final states for the DFA

- All states whose set contain some final state of the NFA

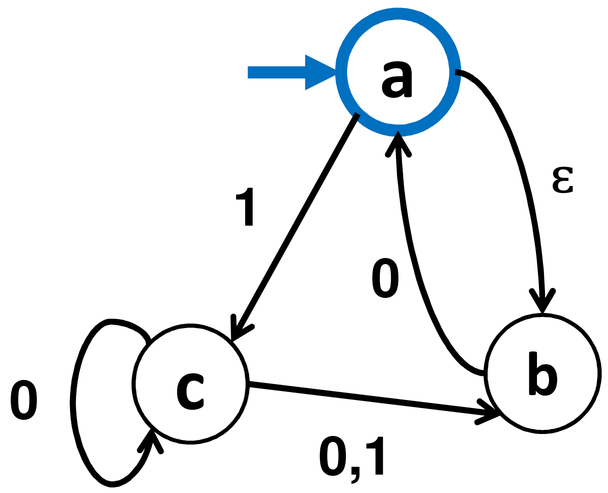


NFA



DFA

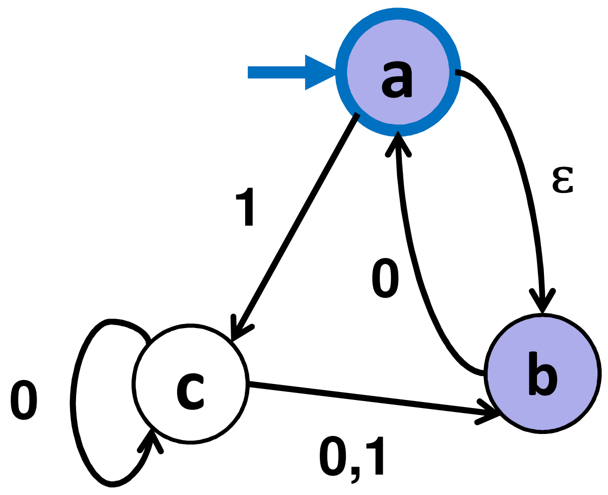
Example: NFA to DFA



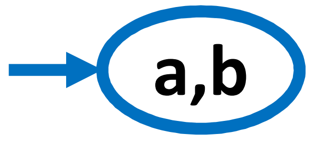
NFA

DFA

Example: NFA to DFA



NFA



DFA

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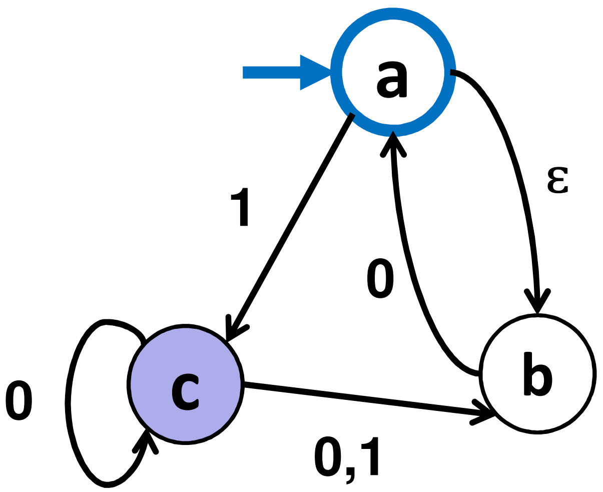
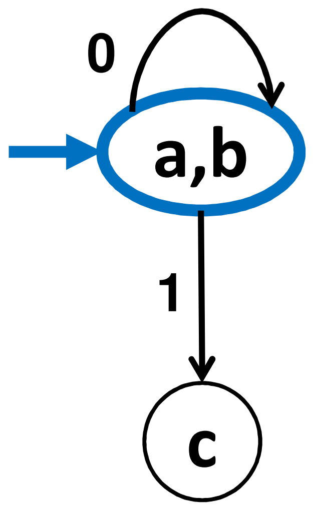
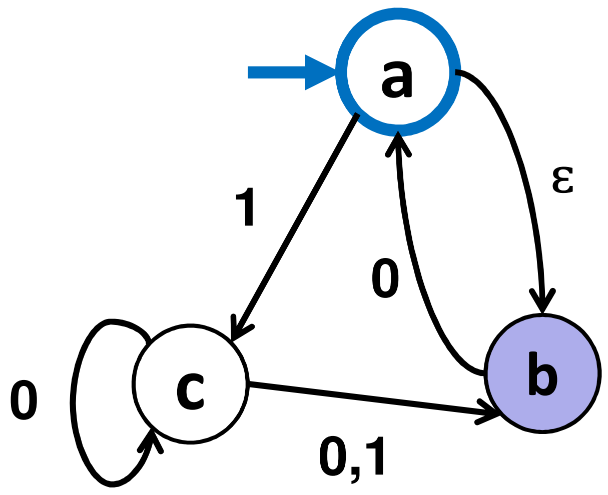


NFA

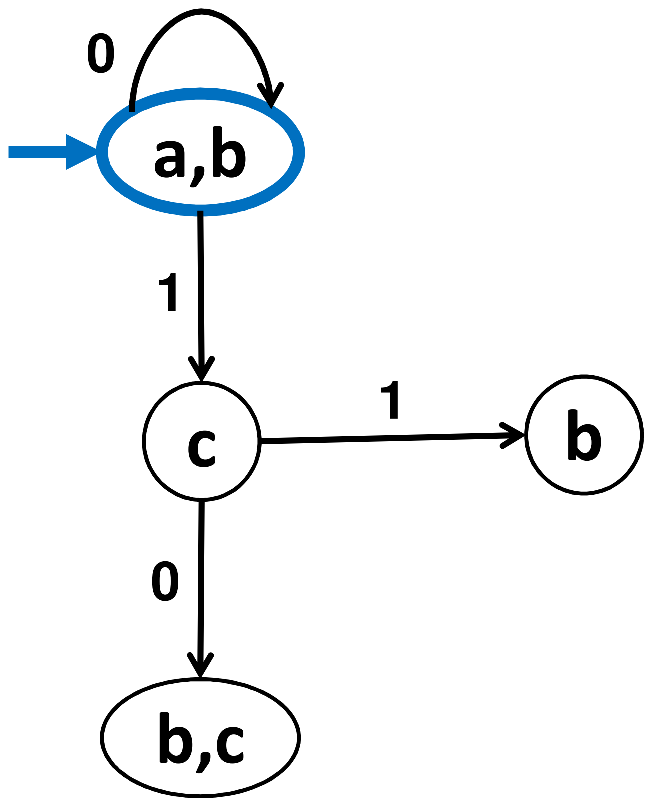


DFA

Example: NFA to DFA

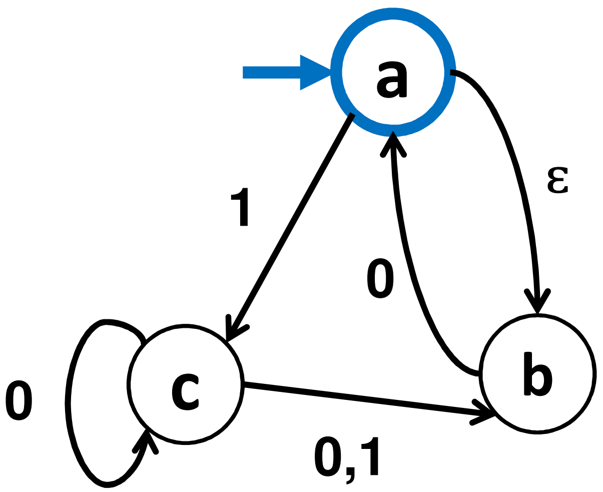


NFA

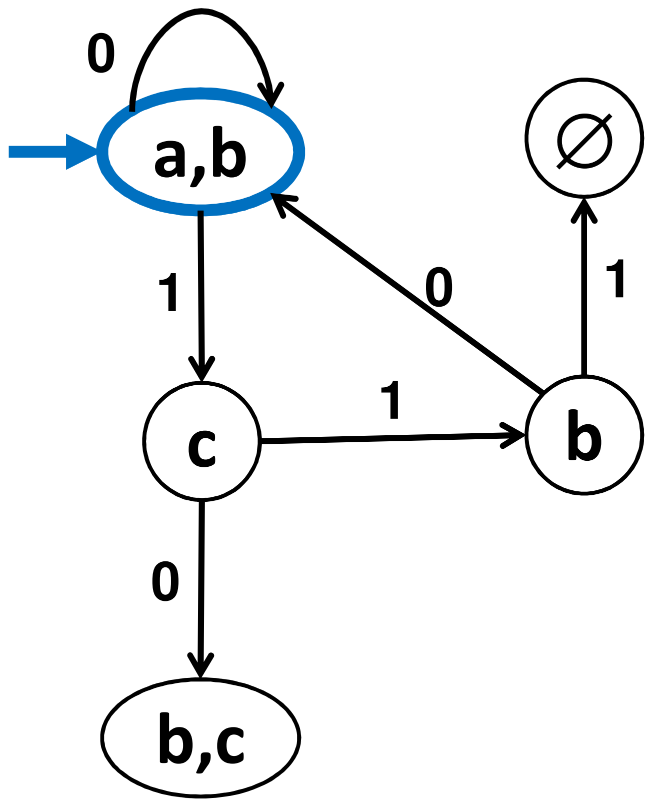


DFA

Example: NFA to DFA

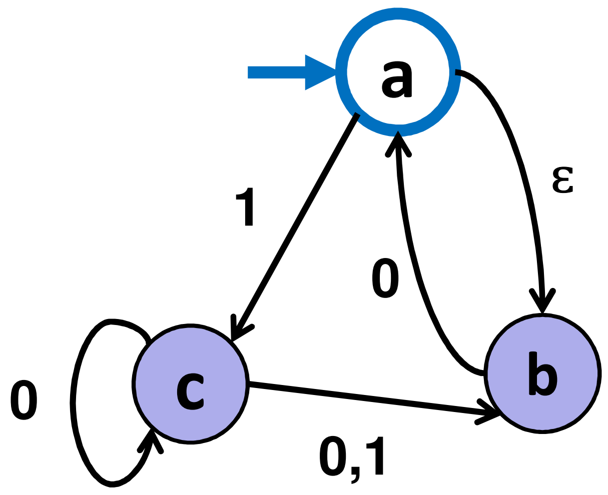


NFA

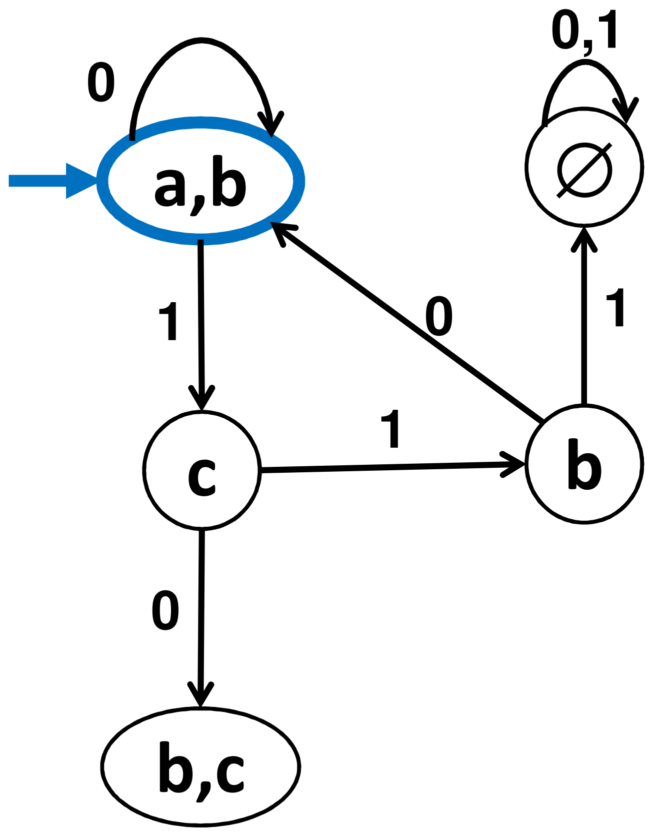


DFA

Example: NFA to DFA

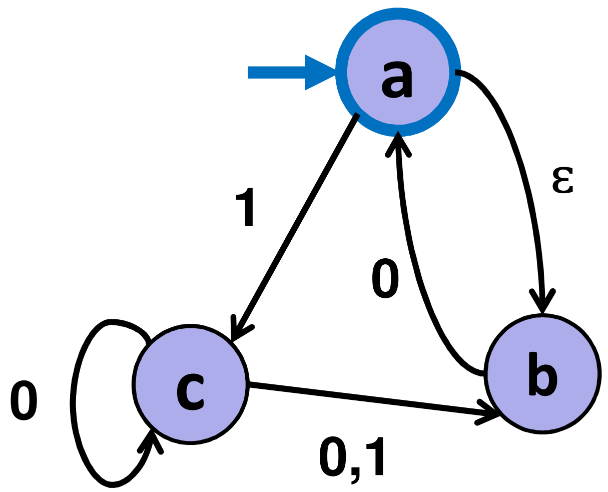


NFA

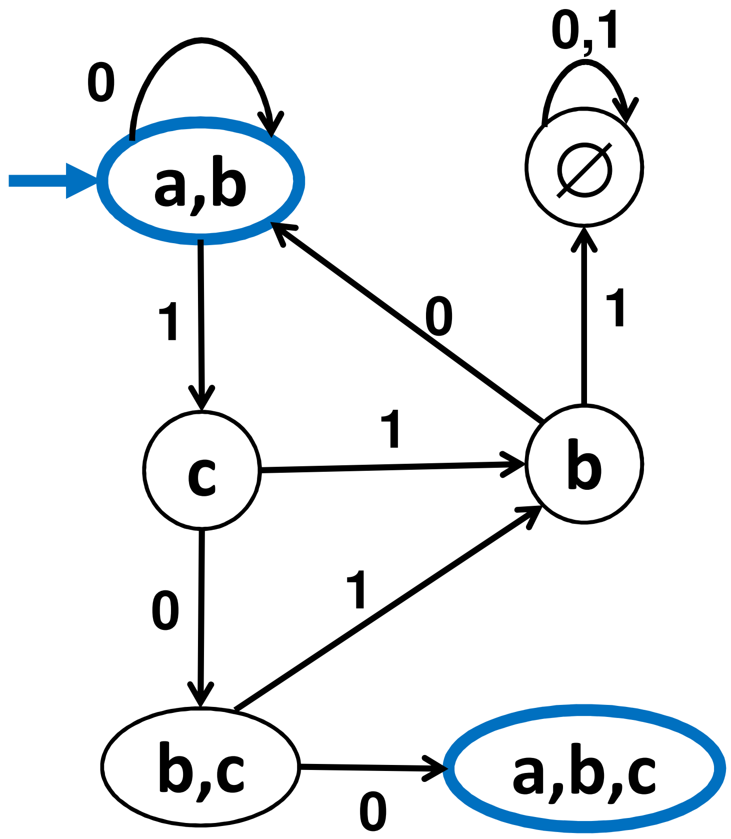


DFA

Example: NFA to DFA

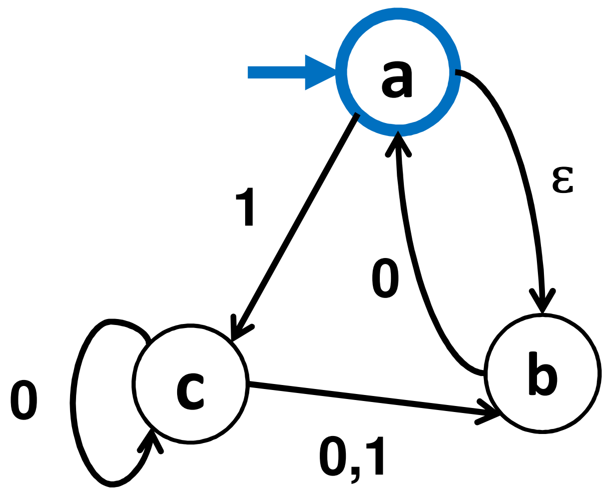


NFA

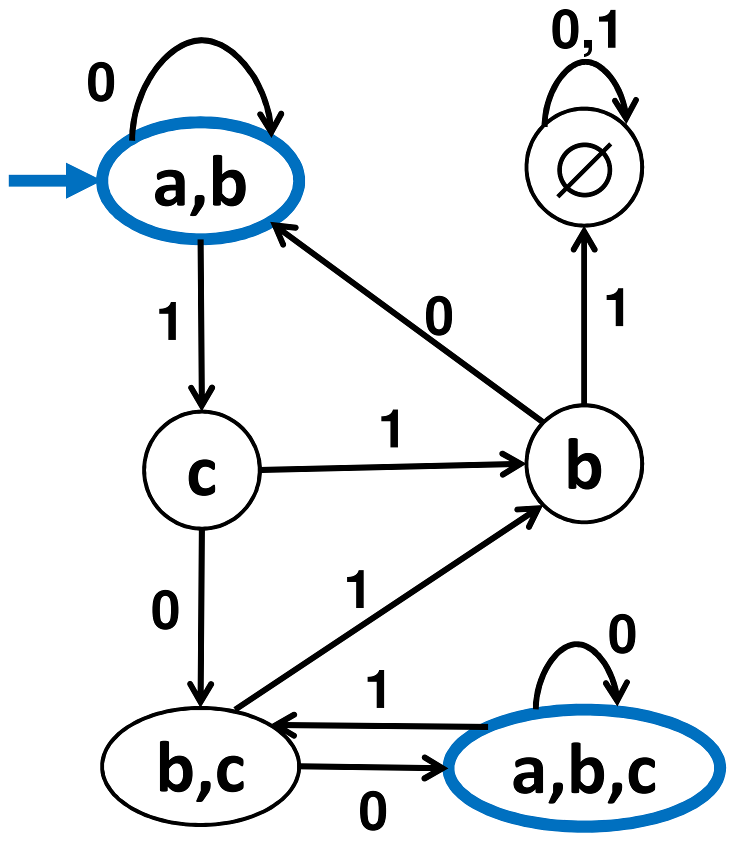


DFA

Example: NFA to DFA

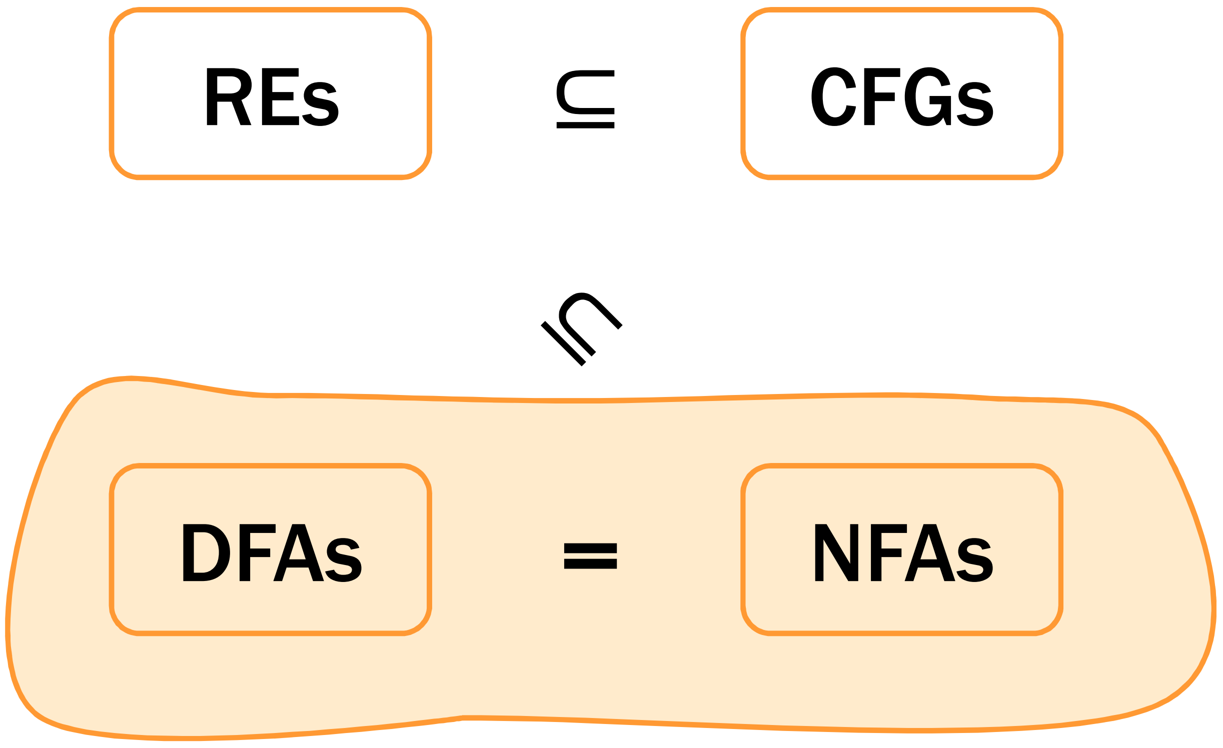


NFA



DFA

The story so far...



Regular expressions ⊆ NFAs ≡ DFAs

We have shown how to build an optimal DFA for every regular expression

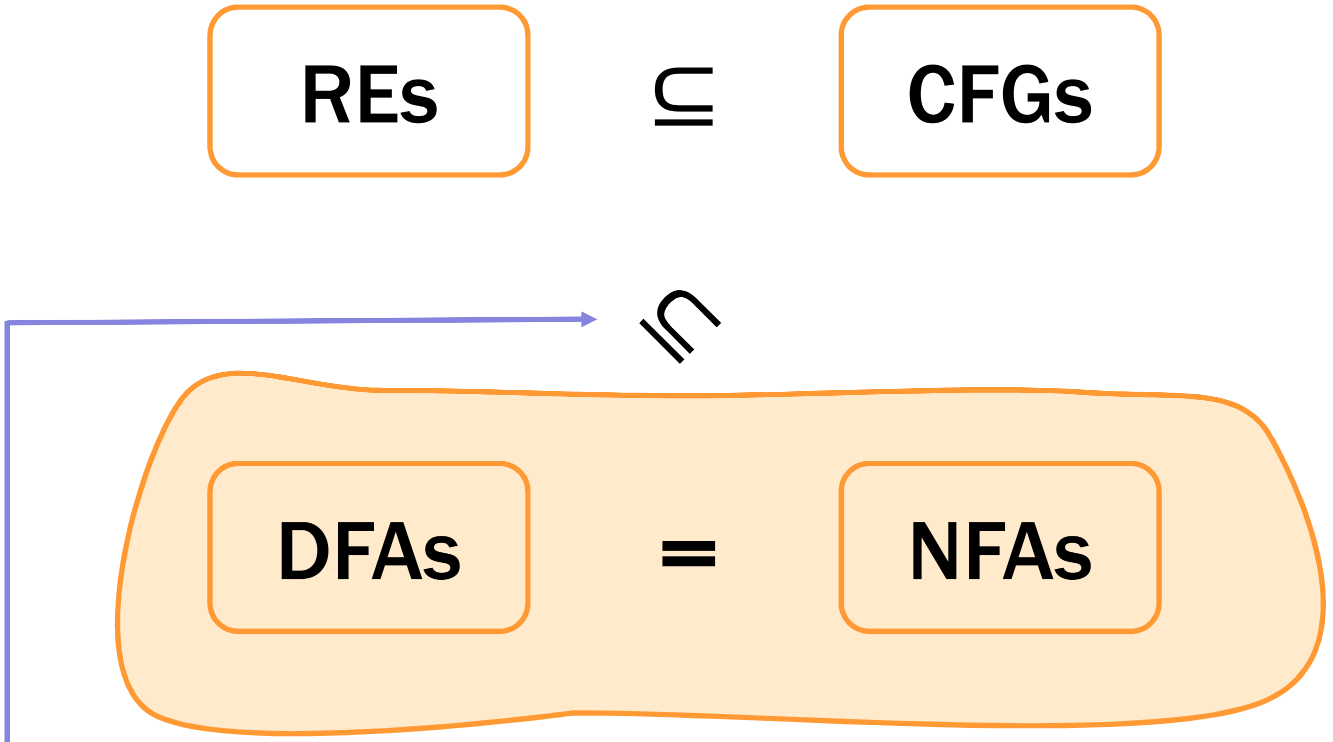
- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Thus, we could now implement a RegExp library

- most RegExp libraries actually simulate the NFA

  by constructing just the parts that are needed during the execution
- (even better: one can combine the two approaches: apply DFA minimization lazily while simulating the NFA)

The story so far...



Is this ⊆ really "=" or "⊊"?

Regular expressions ≡ NFAs ≡ DFAs

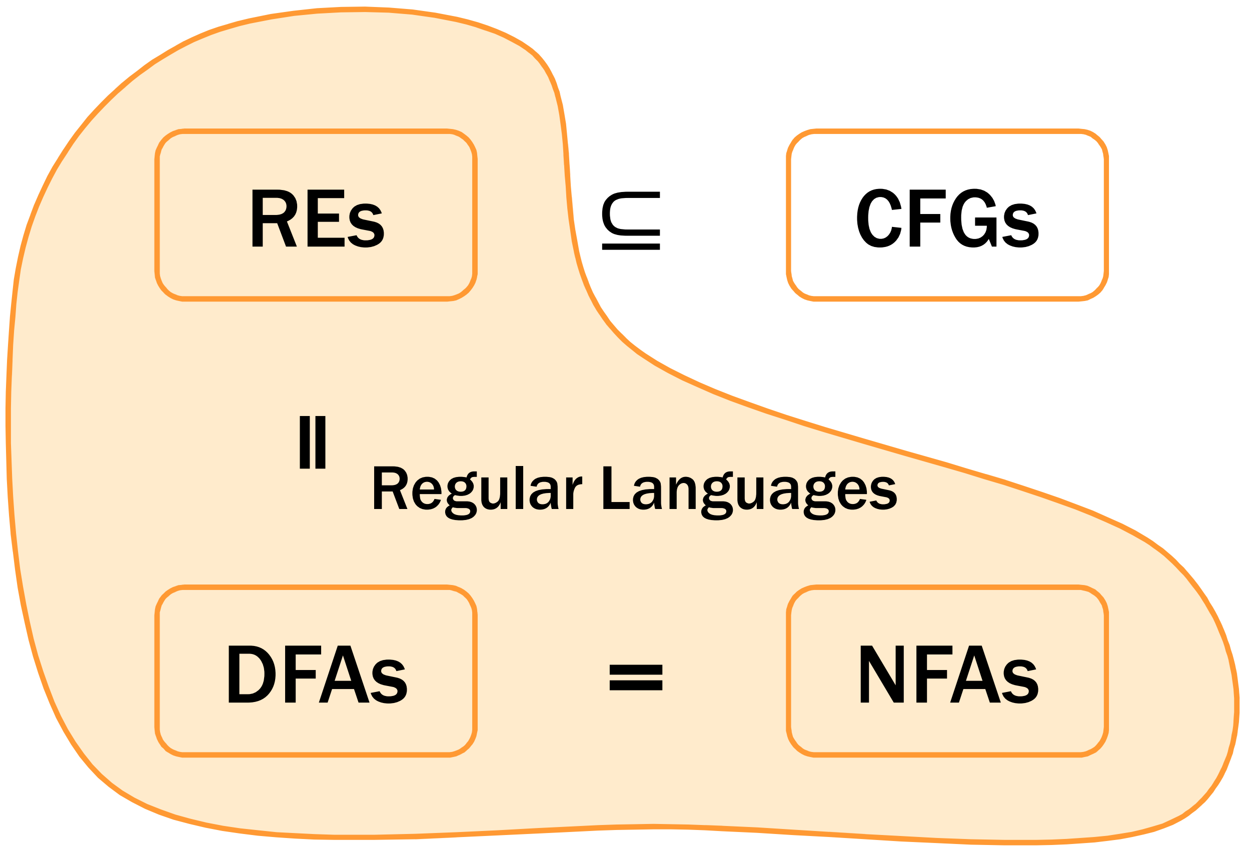
Theorem: For any NFA, there is a regular expression that defines the same language

Corollary: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

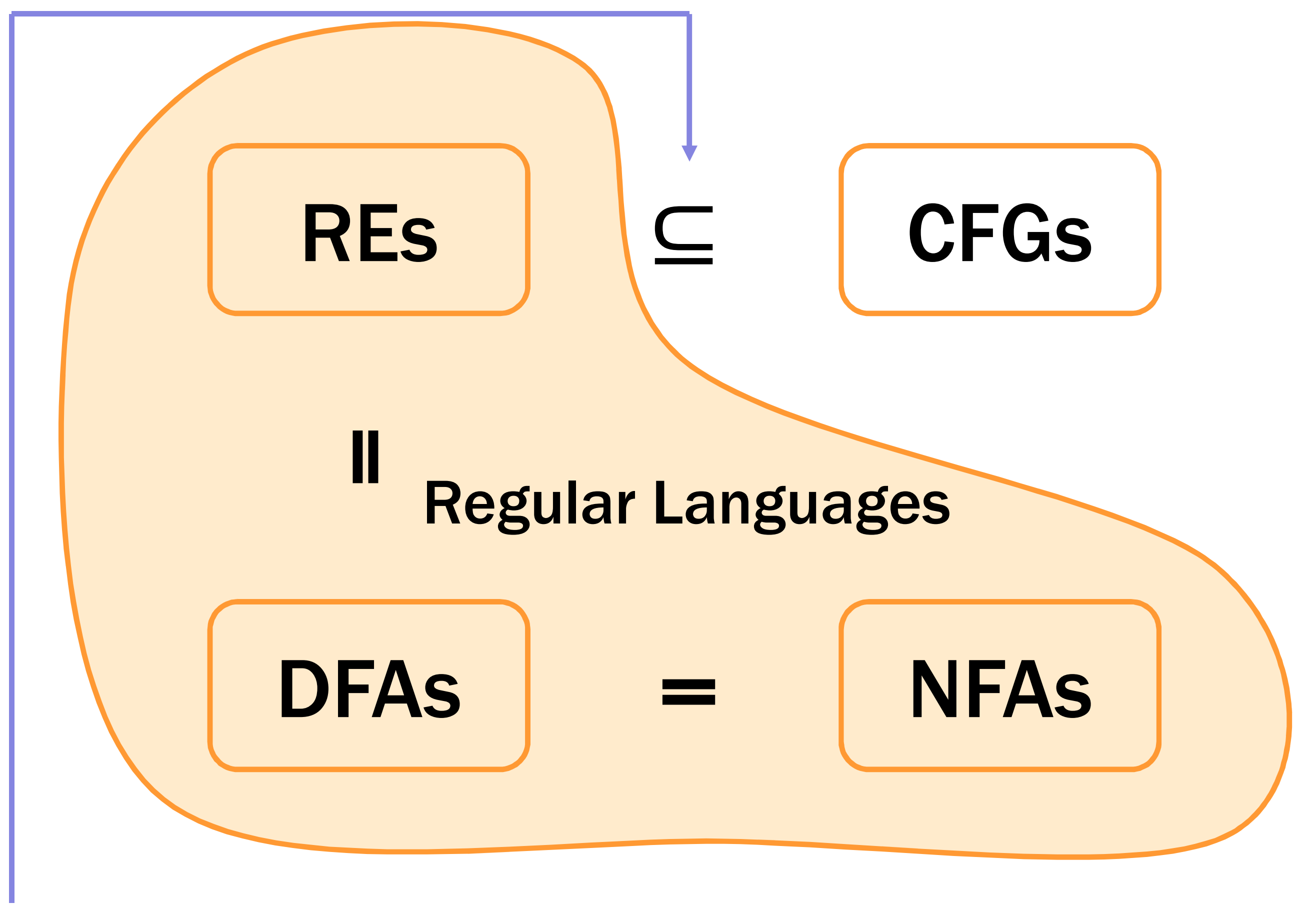
You need to know these facts

- the construction for the Theorem is included in the slides after this, but you will not be tested on it

The story so far...



The story so far...



<u>Next time</u>: Is this ⊆ really "=" or "⊊"?

(Optional) proof that REs ≡ NFAs ≡ DFAs

Theorem: For any NFA, there is a regular expression that defines the same language

Corollary: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

The construction for this Theorem is included in the following slides for your information. You will only need to know the statement of the theorem (and the corollary) not the proof.

We also give an example of the use of this general construction.

New Machinery: Generalized NFAs

- Like NFAs but allow
  - parallel edges (between the same pair of states)
  - regular expressions as edge labels

    NFAs already have edges labeled ε or ***a***

- Machine can follow an edge labeled by A by reading a <u>string of input characters</u> in the language of A
  - (if A is *a* or ε, this matches the original definition, but we now allow REs built with recursive steps.)

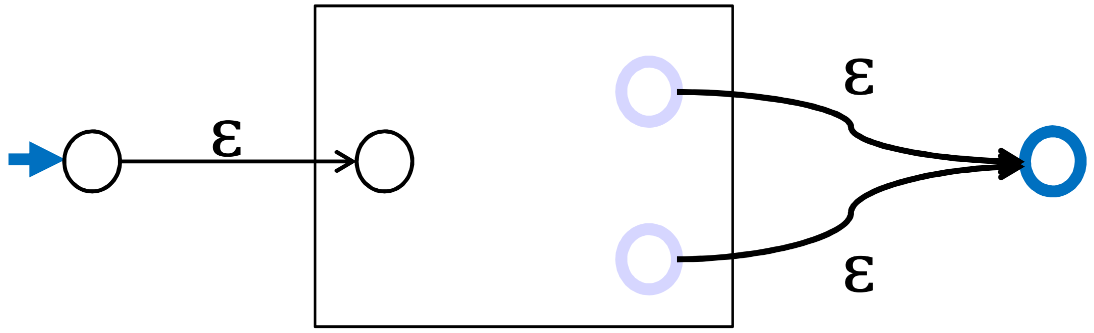
New Machinery: Generalized NFAs

- Like NFAs but allow
  - parallel edges (between the same pair of states)
  - regular expressions as edge labels

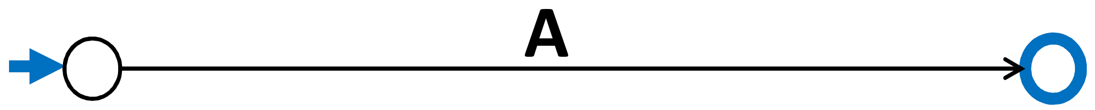
    NFAs already have edges labeled ε or ***a***
- The label of a path is now the concatenation of the *regular expressions* on those edges, making it a regular expression
- Def: A string x is accepted by a generalized NFA iff there is a *path* from start to final state labeled by a regular expression whose language contains x

Construction Idea

Add new start state and final state

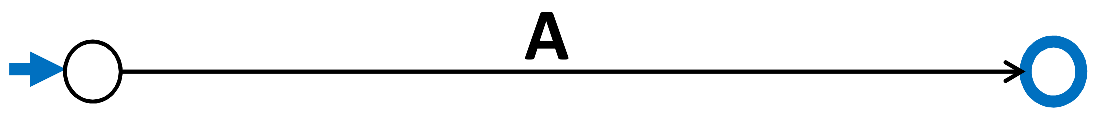


Then delete the original states one by one,
adding edges to keep the same language,
until the graph looks like:



Starting from an NFA

Then delete the original states one by one,
adding edges to keep the same language,
until the graph looks like:



Final graph has only one path to the accepting state,
which is labeled by A,
so it accepts iff x is in the language of A

Thus, A is a regular expression with the same
language as the original NFA.

Only two simplification rules

- **Rule 1:** For any two states q_1 and q_2 with parallel edges (possibly q_1=q_2), replace

$$\begin{tikzcd}
q_1 \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] & q_2 & \text{by} & q_1 \arrow[r, "A \cup B"] & q_2
\end{tikzcd}$$

If the machine would have used the edge labeled A by consuming an input x in the language of A, it can instead use the edge labeled A∪B.

Furthermore, this new edge does not allow transitions for any strings other than those that matched A or B.

Only two simplification rules

- **Rule 2: Eliminate non-start/accepting state q_3 by creating direct edges that skip q_3**

$$\begin{tikzcd} q_1 \arrow[r, "A"] & q_3 \arrow[loop above, "B"] \arrow[r, "C"] & q_2 \end{tikzcd}$$

becomes

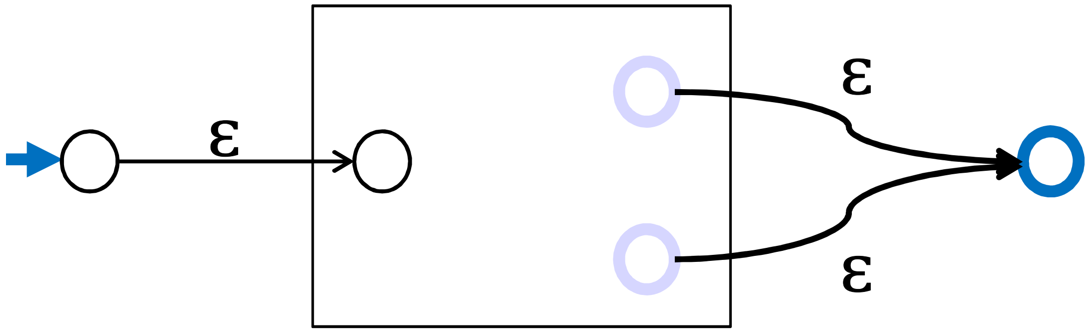
$$\begin{tikzcd} q_1 \arrow[r, "AB^*C"] & q_2 \end{tikzcd}$$

for *every* pair of states q_1, q_2 (even if q_1=q_2)

Any path from q_1 to q_2 would have to match AB^nC for some n (the number of times the self loop was used), so the machine can use the new edge instead. New edge *only* allows strings that were allowed before.

Construction Overview

Add new start state and final state

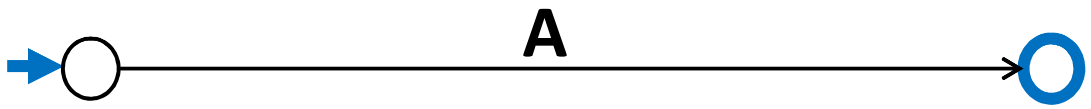


```
While the box contains some state s:
    for all states r, t with (r, s) and (s, t) in E:
        create a direct edge (r, t) by Rule 2
    delete s (no longer needed)
    merge all parallel edges by Rule 1
```

Construction Overview

```
While the box contains some state s:
    for all states r, t with (r, s) and (s, t) in E:
            create a direct edge (r, t) by Rule 2
    delete s (no longer needed)
    merge all parallel edges by Rule 1
```

When the loop exits, the graph looks like this:

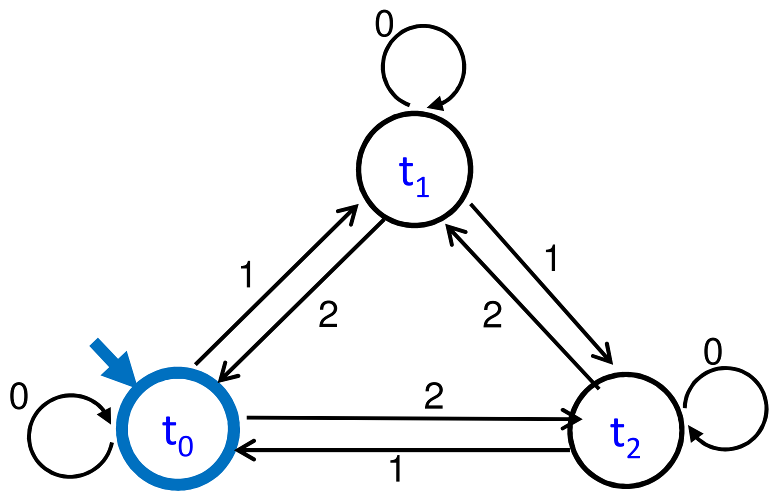


A is a regular expression with the same language as the original NFA.

Converting an NFA to a regular expression

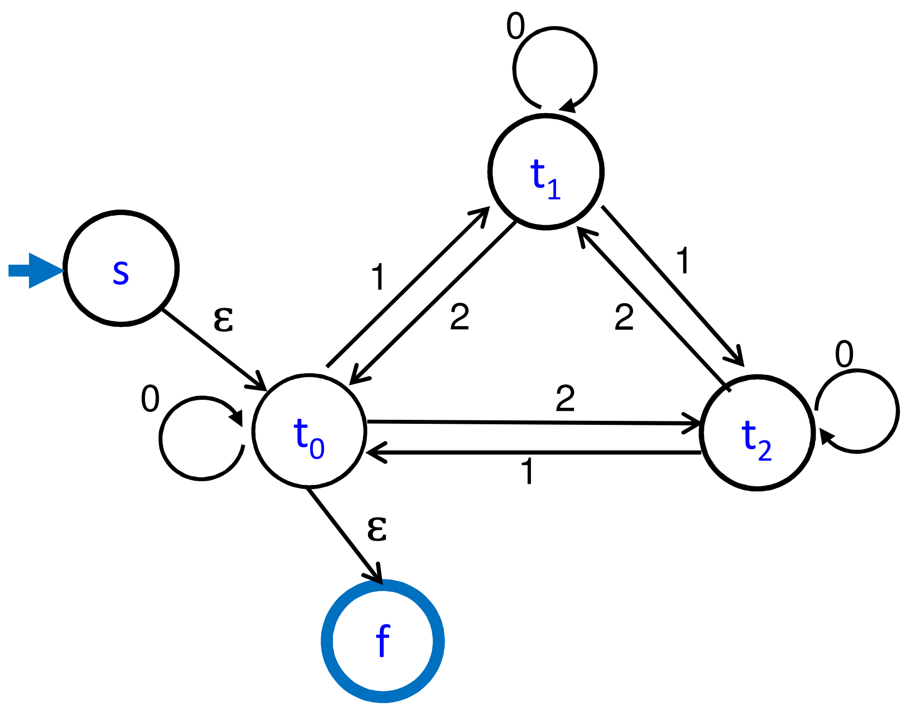
Consider the DFA for the mod 3 sum

– Accept strings from {0,1,2}* where the digits mod 3 sum of the digits is 0



Splicing out a state t_1

Create direct edges between neighbors of t_1 (so that we can delete it afterward)



Splicing out a state t_1

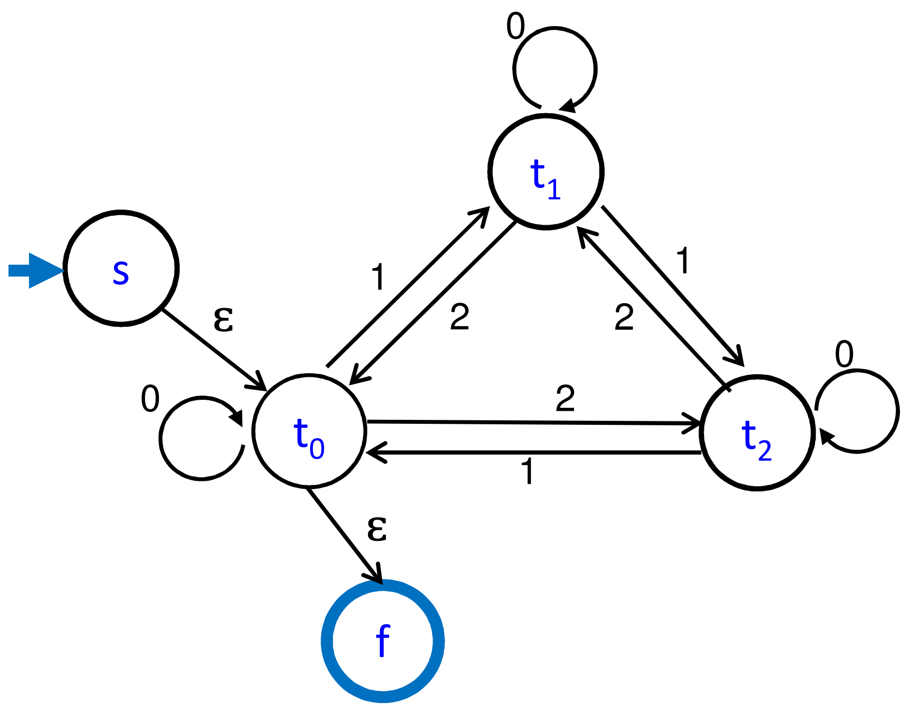
Regular expressions to add to edges

$t_0 \rightarrow t_1 \rightarrow t_0$: 10*2

$t_0 \rightarrow t_1 \rightarrow t_2$: 10*1

$t_2 \rightarrow t_1 \rightarrow t_0$: 20*2

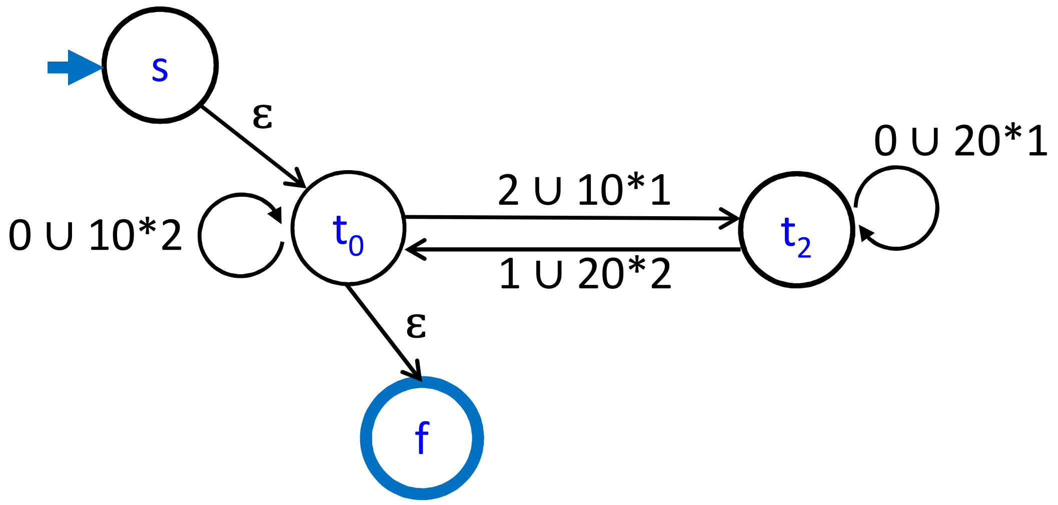
$t_2 \rightarrow t_1 \rightarrow t_2$: 20*1



Splicing out a state t_1

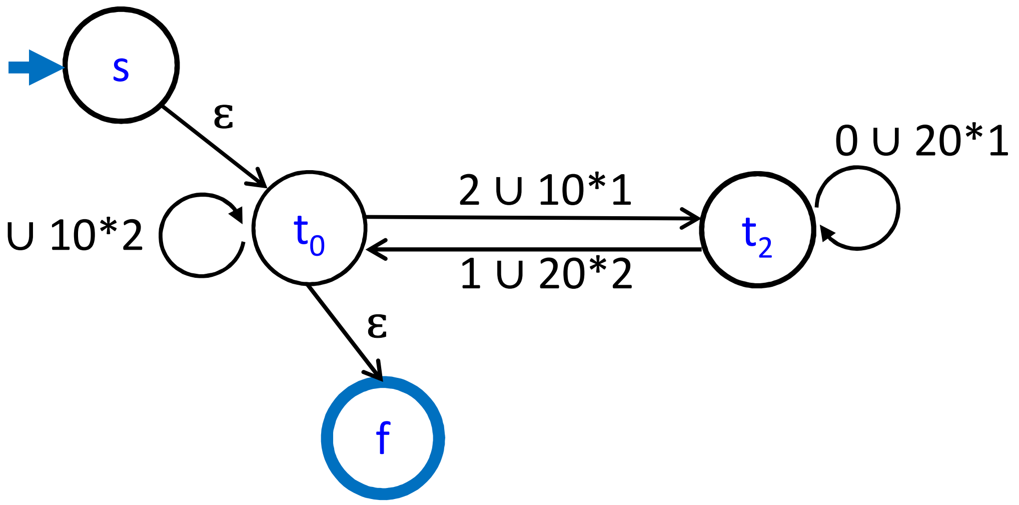
Delete t_1 now that it is redundant

$t_0 \rightarrow t_1 \rightarrow t_0$:	10*2
$t_0 \rightarrow t_1 \rightarrow t_2$:	10*1
$t_2 \rightarrow t_1 \rightarrow t_0$:	20*2
$t_2 \rightarrow t_1 \rightarrow t_2$:	20*1



Splicing out a state t_1

Create direct edges between neighbors of t_2
(so that we can delete it afterward)



Splicing out a state t_1

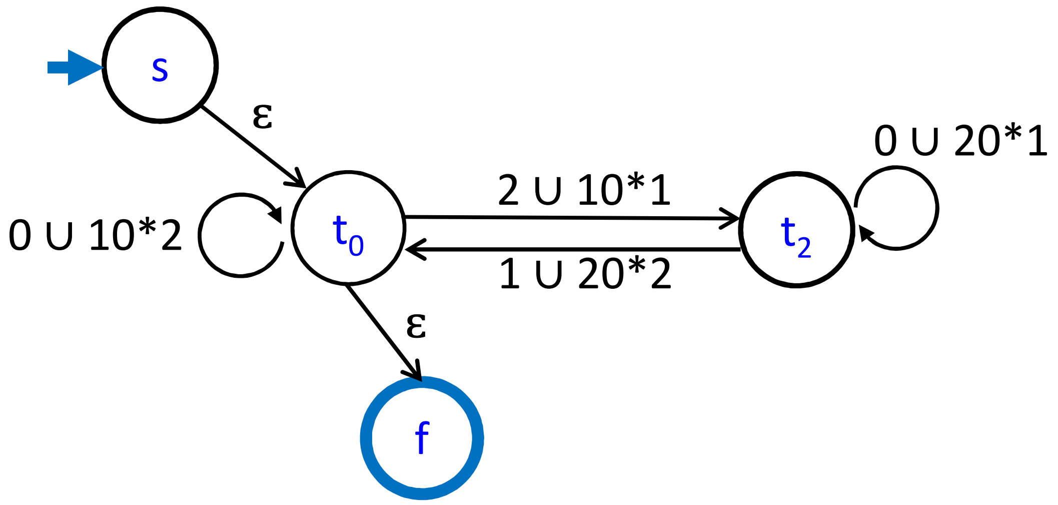
Regular expressions to add to edges

R_1: $0 \cup 10^*2$

R_2: $2 \cup 10^*1$

R_3: $1 \cup 20^*2$

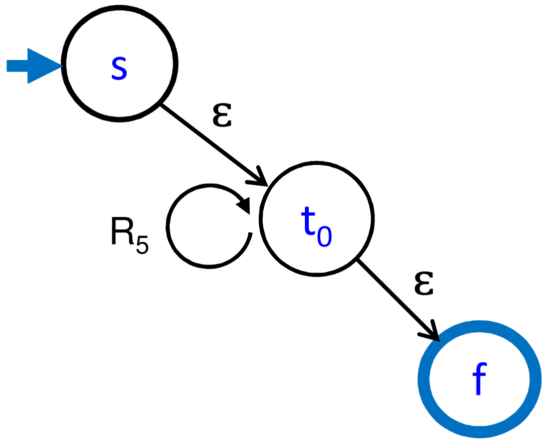
R_4: $0 \cup 20^*1$



Splicing out state t_2 (and then t_0)

Delete t_2 now that it is redundant

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$



R_5: $R_1 \cup R_2R_4{}^*R_3$

Splicing out state t_2 (and then t_0)

Create direct (s,f) edge so we can delete t_0

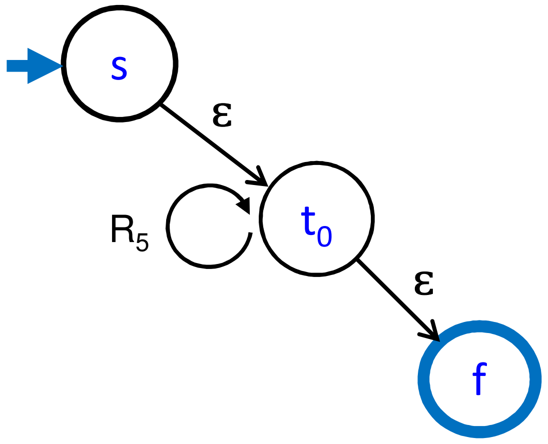
R_1: $0 \cup 10^*2$

R_2: $2 \cup 10^*1$

R_3: $1 \cup 20^*2$

R_4: $0 \cup 20^*1$

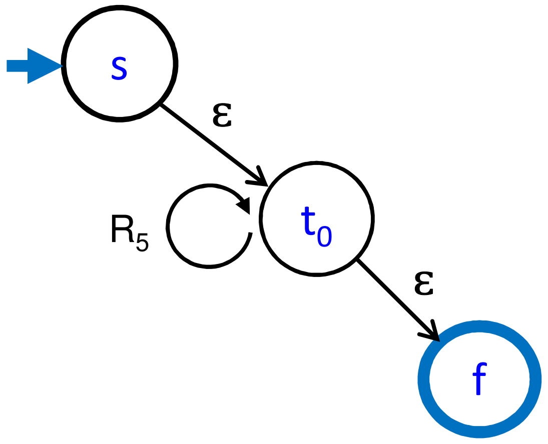
R_5: $R_1 \cup R_2R_4{}^*R_3$



Splicing out state t_2 (and then t_0)

Regular expressions to add to edges

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$

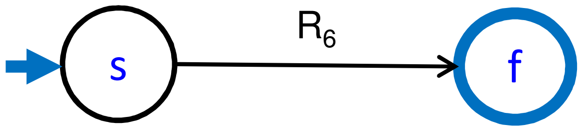


$t_0 \rightarrow t_1 \rightarrow t_0$: $R_5{}^*$

Splicing out state t_2 (and then t_0)

Delete t_0 now that it is redundant

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$

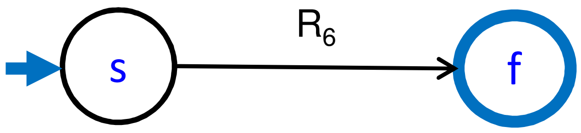


R_6: $R_5{}^*$

Splicing out state t_2 (and then t_0)

Regular expressions to add to edges

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$
R_5: $R_1 \cup R_2R_4{}^*R_3$
R_6: $R_5{}^*$



Final regular expression: R_6 =
$(0 \cup 10^*2 \cup (2 \cup 10^*1)(0 \cup 20^*1)^*(1 \cup 20^*2))^*$