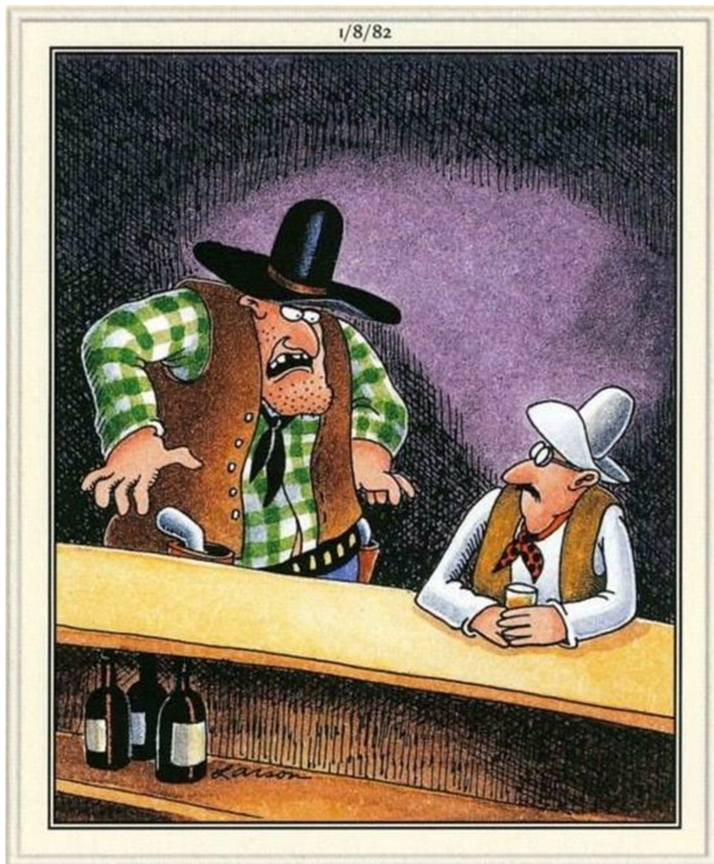
CSE 311: Foundations of Computing

Lecture 12: Modular Exponentiation, Set Theory



“I *asked* you a question, buddy. ... What’s the square root of 5,248?”

Last class: Euclid's Algorithm for GCD

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(a, 0) = a$.

Check out video on Section 4 solutions

Equations with recursive calls:

gcd(660,126) = gcd(126, 660 mod 126) = gcd(126, 30)
= gcd(30, 126 mod 30) = gcd(30, 6)
= gcd(6, 30 mod 6) = gcd(6, 0)
= 6

Tableau form (which is much easier to work with and will be more useful):

660 = 5 * 126 + 30
126 = 4 * 30 + 6
30 = 5 * 6 + 0

Each line computes both quotient and remainder of the shifted numbers

Last class: Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Example: $a = 35, b = 27$

Compute $\gcd(35, 27)$:

$$\begin{aligned}35 &= 1 * 27 + 8 \\ 27 &= 3 * 8 + 3 \\ 8 &= 2 * 3 + 2 \\ 3 &= 1 * 2 + 1 \\ 2 &= 2 * 1 + 0\end{aligned}$$

$$\begin{aligned}8 &= 35 - 1 * 27 \\ 3 &= 27 - 3 * 8 \\ 2 &= 8 - 2 * 3 \\ 1 &= 3 - 1 * 2\end{aligned}$$

Last class: Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Example: $a = 35, b = 27$

Use equations to substitute back

$$8 = 35 - 1 * 27$$
$$3 = 27 - 3 * 8$$
$$2 = 8 - 2 * 3$$
$$1 = 3 - 1 * 2$$

$$\begin{aligned}
1 &= 3 - 1 * 2 \\
&= 3 - 1 * (8 - 2 * 3) \\
&= 3 - 8 + 2 * 3 \\
&= (-1) * 8 + 3 * 3 \\
&= (-1) * 8 + 3 * (27 - 3 * 8) \\
&= (-1) * 8 + 3 * 27 + (-9) * 8 \\
&= 3 * 27 + (-10) * 8 \\
&= 3 * 27 + (-10) * (35 - 1 * 27) \\
&= 3 * 27 + (-10) * 35 + 10 * 27 \\
&= (-10) * 35 + 13 * 27
\end{aligned}$$

Optional Check:

$$(-10) * 35 = -350$$
$$13 * 27 = 351$$

Last class: Multiplicative inverse $(\mathrm{mod}\ m)$

Let $0 \le a, b < m$. Then, b is the *multiplicative inverse of* a *(modulo* m*)* iff $ab \equiv 1 \pmod{m}$.

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

mod 7

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

mod 10

Last class: Multiplicative inverse $(\mathrm{mod}\ m)$

Let $0 \le a, b < m$. Then, b is the *multiplicative inverse of* a *(modulo* m*)* iff $ab \equiv 1 \pmod{m}$.

This can't exist if a and m have a common factor >1.

Idea: b is like $a^{-1} \pmod{m}$ so multiplying by b is equivalent to dividing by a.

x	0	**1**	2	**3**	4	5	6	**7**	8	**9**
0	0	0	0	0	0	0	0	0	0	0
1	0	**1**	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	**1**	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	**1**	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	**1**

mod 10

Finding multiplicative inverse $\bmod m$

Suppose that $\gcd(a, m) = 1$.

Using Extended Euclidean Algorithm

find integers s and t such that $sa + tm = 1$.

Therefore $sa \equiv 1 \pmod{m}$.

The multiplicative inverse b of a modulo m must also satisfy $0 \leq b < m$ so we set $b = s \bmod m$.

It works since $ba \equiv sa \equiv 1 \pmod{m}$

Example

Solve: $7x \equiv 1 \pmod{26}$

Example

Solve: $7x \equiv 1 \pmod{26}$

First compute and check that $\gcd(26, 7) = 1$

$$26 = 3 * 7 + 5$$
$$7 = 1 * 5 + 2$$
$$5 = 2 * 2 + 1$$
$$2 = 2 * 1 + 0$$

Example

Solve: $7x \equiv 1 \pmod{26}$

Then rewrite equations in form for substitution

$$\begin{aligned} 26 &= 3 * 7 + 5 \\ 7 &= 1 * 5 + 2 \\ 5 &= 2 * 2 + 1 \\ 2 &= 2 * 1 + 0 \end{aligned} \qquad \begin{aligned} 5 &= 26 - 3 * 7 \\ 2 &= 7 - 1 * 5 \\ 1 &= 5 - 2 * 2 \end{aligned}$$

Example

Solve: $7x \equiv 1 \pmod{26}$

Apply substitutions from bottom to top.

$$26 = 3 * 7 + 5 \qquad 5 = 26 - 3 * 7$$

$$7 = 1 * 5 + 2 \qquad 2 = 7 - 1 * 5$$

$$5 = 2 * 2 + 1 \qquad 1 = 5 - 2 * 2$$

$$2 = 2 * 1 + 0$$

$$\begin{aligned} 1 &= 5 - 2 * 2 \\ &= 5 - 2 * (7 - 1 * 5) \\ &= (-2) * 7 + 3 * 5 \\ &= (-2) * 7 + 3 * (26 - 3 * 7) \\ &= (-11) * 7 + 3 * 26 \end{aligned}$$

Example

Solve: $7x \equiv 1 \pmod{26}$

Read off coefficient and reduce modulo 26.

$$\begin{aligned} 26 &= 3*7 + 5 \qquad & 5 &= 26 - 3*7 \\ 7 &= 1*5 + 2 & 2 &= 7 - 1*5 \\ 5 &= 2*2 + 1 & 1 &= 5 - 2*2 \\ 2 &= 2*1 + 0 \end{aligned}$$

$$\begin{aligned} 1 &= 5 - 2*2 \\ &= 5 - 2*(7 - 1*5) \\ &= (-2)*7 + 3*5 \\ &= (-2)*7 + 3*(26 - 3*7) \\ &= (-11)*7 + 3*26 \end{aligned}$$

Multiplicative inverse of 7 modulo 26

Now $(-11) \bmod 26 = 15$. **So,** $x = 15 + 26k$ **for integer** k.

Example of a more general equation

7^{-1} 15

Now solve: $7y \equiv 3 \pmod{26}$

We already computed that 15 is the multiplicative inverse of 7 modulo 26. That is, $7 \cdot 15 \equiv 1 \pmod{26}$

If y is a solution, then multiplying by 15 we have

$$15 \cdot 7 \cdot y \equiv 15 \cdot 3 \pmod{26}$$

15
$\times 3$
45

Substituting $15 \cdot 7 \equiv 1 \pmod{26}$ on the left gives

$$y = 1 \cdot y \equiv 15 \cdot 3 \equiv 19 \pmod{26}$$

This shows that <u>every</u> solution y is congruent to 19.

Example of a more general equation

Now solve: $7y \equiv 3 \pmod{26}$

Multiplying both sides of $y \equiv 19 \pmod{26}$ by 7 gives

$$7y \equiv 7 \cdot 19 \equiv 3 \pmod{26}$$

So, any $y \equiv 19 \pmod{26}$ is a solution.

Thus, the set of numbers of the form $y = 19 + 26k$, for any integer k, are exactly solutions of this equation.

Math mod a prime is especially nice

$\gcd(a, m) = 1$ **if** m **is prime and** $0 < a < m$ **so can always solve these equations mod a prime.**

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0	0	0	0	0
1	0	**1**	2	3	4	5	6
2	0	2	4	6	**1**	3	5
3	0	3	6	2	5	**1**	4
4	0	4	**1**	5	2	6	3
5	0	5	3	**1**	6	4	2
6	0	6	5	4	3	2	**1**

mod 7

Hashing

Scenario:

Map a small number of data values from a large domain $\{0, 1, \dots, M-1\}$ …

…into a small set of locations $\{0,1, \dots, n-1\}$ so one can quickly check if some value is present

- $\mathrm{hash}(x) = (ax + b) \bmod p$ for p a prime close to n
  - Relies on $\gcd(a, p) = 1$ to avoid many collisions
- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

Hashing

- $\mathrm{hash}(x) = (ax + b) \bmod p$ for p a prime close to n

- Applications
  - map integer to location in array (hash tables)
  - map user ID or IP address to machine

    requests from the same user / IP address go to the same machine

    requests from different users / IP addresses spread randomly

Attack on RSA security with GCD

- RSA *public key* includes m that is the product of two large *randomly chosen* primes p, q
  - Everyone can see all the public keys (millions)
  - Security depends on keeping p and q secret
  - OK since factoring m seems very hard
- In 2012 a new attack using GCD broke a huge number of RSA public keys!
  - Weak keys: Algorithms/devices cut corners: Skimped on random bits or size of primes

Attack on RSA security with GCD

Weak keys: few random bits

- Few enough that some public keys m_1 and m_2 happen to share just one of their two factors:

$$m_1 = pq \text{ and } m_2 = pr$$

- Then can break both since $p = \gcd(m_1, m_2)$

2012: 11 million RSA keys, 23,500 broken

2016: 1024-bit RSA keys available from Internet

- 26 million keys, 63,500 broken

2019: 750 million RSA keys, 250,000 broken

- IoT (Internet of Things) devices often the culprit

RSA Relies on Modular Exponentiation

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1	1	1	1	1	1	1
2	2	4	1	2	4	1
3	3	2	6	4	5	1
4	4	2	1	4	2	1
5	5	4	6	2	3	1
6	6	1	6	1	6	1

7

mod 7

Modular Exponentiation: (Essential for RSA)

- **Compute** 78365^{81453}

  How many digits? = 5 x 81453 digits long

- **Compute** 78365^{81453} mod 104729

- **Output is small**
  - need to keep intermediate results small

Small Multiplications

$a = qm + r$ remainder when we divide by m

By the multiplicative property modulo $\boldsymbol{m}$, if you want to compute $\boldsymbol{ab} \bmod \boldsymbol{m}$ then you can do the following:

1. Reduce $\boldsymbol{a}$ and $\boldsymbol{b}$ modulo $\boldsymbol{m}$ to get $\boldsymbol{a} \bmod \boldsymbol{m}$ and $\boldsymbol{b} \bmod \boldsymbol{m}$
2. Multiply to produce $\boldsymbol{c} = (\boldsymbol{a} \bmod \boldsymbol{m})(\boldsymbol{b} \bmod \boldsymbol{m})$
3. Output $\boldsymbol{c} \bmod \boldsymbol{m}$

Claim: $\boldsymbol{c} \bmod \boldsymbol{m} = \boldsymbol{ab} \bmod \boldsymbol{m}$

Proof: Just need to show that $\boldsymbol{c} \equiv \boldsymbol{ab} \pmod{\boldsymbol{m}}$.

That follows from $(\boldsymbol{a} \bmod \boldsymbol{m}) \equiv \boldsymbol{a} \pmod{\boldsymbol{m}}$

$(\boldsymbol{b} \bmod \boldsymbol{m}) \equiv \boldsymbol{b} \pmod{\boldsymbol{m}}$

and the multiplicative property since $\boldsymbol{c}$ is the product of the left sides and $\boldsymbol{ab}$ is the product of the right sides. ■

Repeated Squaring – small and fast

Then we have $\boldsymbol{ab} \bmod \boldsymbol{m} = \left((\boldsymbol{a} \bmod \boldsymbol{m})(\boldsymbol{b} \bmod \boldsymbol{m})\right) \bmod \boldsymbol{m}$

So $\quad \boldsymbol{a}^2 \bmod \boldsymbol{m} = (\boldsymbol{a} \bmod \boldsymbol{m})^2 \bmod \boldsymbol{m}$ 1

and $\quad \boldsymbol{a}^4 \bmod \boldsymbol{m} = (\boldsymbol{a}^2 \bmod \boldsymbol{m})^2 \bmod \boldsymbol{m}$ 2

and $\quad \boldsymbol{a}^8 \bmod \boldsymbol{m} = (\boldsymbol{a}^4 \bmod \boldsymbol{m})^2 \bmod \boldsymbol{m}$ 3

and $\quad \boldsymbol{a}^{16} \bmod \boldsymbol{m} = (\boldsymbol{a}^8 \bmod \boldsymbol{m})^2 \bmod \boldsymbol{m}$ 4

and $\quad \boldsymbol{a}^{32} \bmod \boldsymbol{m} = (\boldsymbol{a}^{16} \bmod \boldsymbol{m})^2 \bmod \boldsymbol{m}$ 5

2

Can compute $\boldsymbol{a}^{k} \bmod \boldsymbol{m}$ for $\boldsymbol{k} = \mathbf{2}^{i}$ in only $\boldsymbol{i}$ steps

What if $\boldsymbol{k}$ is not a power of $\mathbf{2}$?

Fast Exponentiation Algorithm

81453 in binary is 10011101000101101

$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$

$$a^{81453} = a^{2^{16}+2^{13}+2^{12}+2^{11}+2^{9}+2^{5}+2^{3}+2^{2}+2^{0}}$$
$$= a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$$

$$a^{81453} \bmod m = (a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}) \bmod m$$

$$= (\ldots(((((a^{2^{16}} \bmod m \cdot$$
$$a^{2^{13}} \bmod m\ ) \bmod m \cdot$$
$$a^{2^{12}} \bmod m) \bmod m \cdot$$
$$a^{2^{11}} \bmod m) \bmod m \cdot$$
$$a^{2^{9}} \bmod m) \bmod m \cdot$$
$$a^{2^{5}} \bmod m) \bmod m \cdot$$
$$a^{2^{3}} \bmod m) \bmod m \cdot$$
$$a^{2^{2}} \bmod m) \bmod m \cdot$$
$$a^{2^{0}} \bmod m) \bmod m$$

Uses only 16 + 8 = 24 multiplications

See new slide added to "slides"

$a^{10} \bmod m$

$10 = (1010)_2 = 2^3 + 2^1$

a^2, a^4, a^8

The fast exponentiation algorithm computes $a^k \bmod m$ using $\leq 2\log k$ multiplications $\bmod m$

Fast Exponentiation: $a^k \bmod m$ for all k

Another way….

$$a^{2j} \bmod m = \left(a^j \bmod m\right)^2 \bmod m$$

$$a^{2j+1} \bmod m = \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m$$

Recursive Fast Exponentiation

```
public static int FastModExp(int a, int k, int modulus) {

        if (k == 0) {
            return 1;

        } else if ((k % 2) == 0) {
            long temp = FastModExp(a,k/2,modulus);
            return (temp * temp) % modulus;

        } else {
            long temp = FastModExp(a,k-1,modulus);
            return (a * temp) % modulus;
        }
}
```

$$a^{2j} \bmod m = \left(a^{j} \bmod m\right)^{2} \bmod m$$

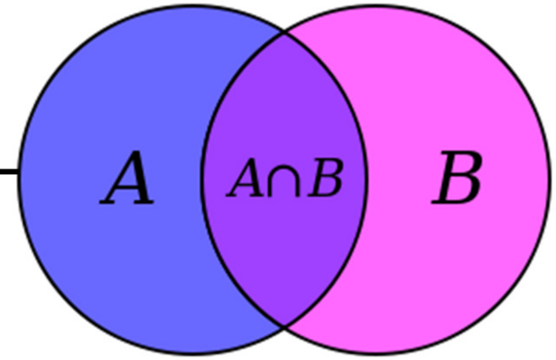
$$a^{2j+1} \bmod m = \big((a \bmod m) \cdot (a^{2j} \bmod m)\big) \bmod m$$

Using Fast Modular Exponentiation

- Your e-commerce web transactions use SSL (Secure Socket Layer) based on RSA encryption
- RSA ...as of 2023
  - Vendor chooses random 1024-bit or 2048-bit primes p, q and 1024/2048-bit exponent e. Computes $m = p \cdot q$
  - Vendor broadcasts (m, e)
  - To send a to vendor, you compute $C = a^e \bmod m$ using *fast modular exponentiation* and send C to the vendor.
  - Using secret p, q the vendor computes d that is the *multiplicative inverse* of e mod $(p-1)(q-1)$.
  - Vendor computes $C^d \bmod m$ using *fast modular exponentiation*.
  - Fact: $a = C^d \bmod m$ for $0 < a < m$ unless $p|a$ or $q|a$

Sets

Sets



Sets are collections of objects called elements.

Write $a \in B$ to say that a is an element of set B, and $a \notin B$ to say that it is not.

Some simple examples

$A = \{1\}$

$B = \{1, 3, 2\}$

$C = \{\square, 1\}$

$D = \{\{17\}, 17\}$

$E = \{1, 2, 7, \text{cat}, \text{dog}, \varnothing, \alpha\}$

Some Common Sets

$\mathbb{N}$

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N}$ = {0, 1, 2, …}

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z}$ = {…, -2, -1, 0, 1, 2, …}

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π, $\sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a natural number $n \geq 0$

∅ = **{}** is the **empty set**; the *only* set with no elements

$\mathbb{Q}$ quotient

$\mathbb{Z}$ Z zahlen

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\varnothing\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \;:=\; \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;:=\; \forall x\,(x \in A \rightarrow x \in B)$$

- Notes:

$$(A = B) \equiv (A \subseteq B) \wedge (B \subseteq A)$$

$A \supseteq B$ means $B \subseteq A$

proper subset

$A \subset B$ means $A \subseteq B$ but $A \neq B$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \ := \ \forall x\,(x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal to each other?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \ := \ \forall x\, (x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

QUESTIONS

$\varnothing \subseteq A$? ✓

$A \subseteq B$? ✗

$C \subseteq B$? ✓

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \;:=\; \forall\, x\,(x \in A \rightarrow x \in B)$$

Note the domain restriction.

We will use a shorthand restriction to a set

$$\forall x \in A,\, P(x) \;:=\; \forall x\,(x \in A \rightarrow P(x))$$

Restricting all quantified variables improves *clarity*