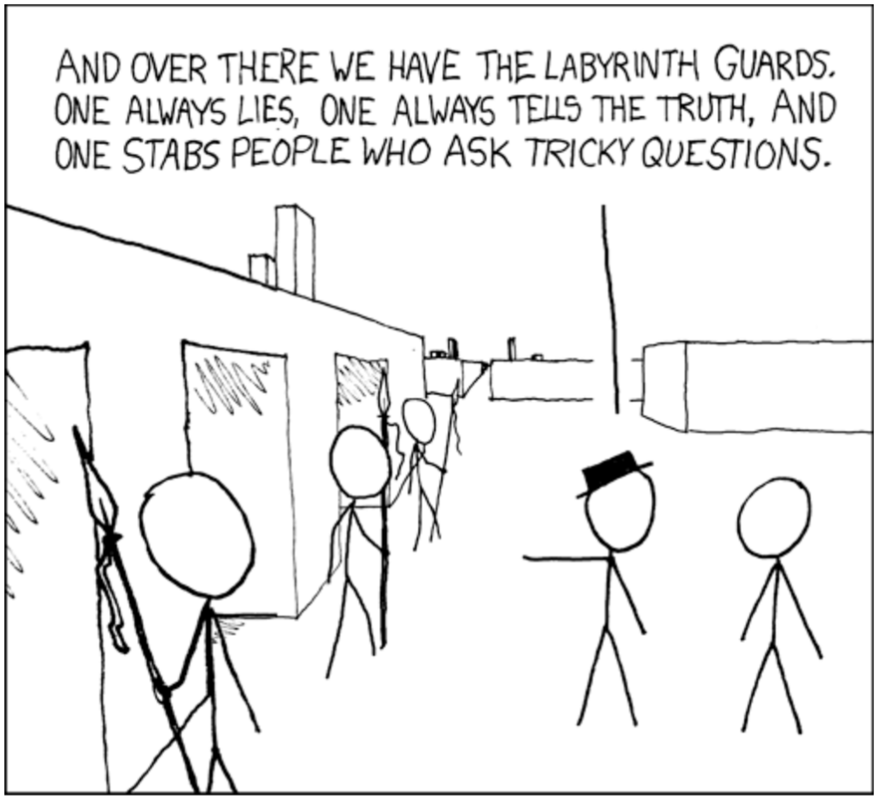
CSE 311: Foundations of Computing

Lecture 6: Predicate Logic, Logical Inference



Last Class: Quantifiers

We use *quantifiers* to talk about collections of objects.

$\forall x\ P(x)$

P(x) is true **for every** x in the domain

read as "**for all x, P of x**"

$\exists x\ P(x)$

There is an x in the domain for which P(x) is true

read as "**there exists x, P of x**"

Last class: Predicate Logic to English (Natural)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Translate the following statements to English

$\forall x\, \exists y$ Greater(y, x)

For every positive integer, there is a larger positive integer.

$\exists y\, \forall x$ Greater(y, x)

There is a positive integer that is larger than every other positive integer.

$\forall x\, \exists y$ (Greater(y, x) $\wedge$ Prime(y))

For every positive integer, there is a prime that is larger.

Sound more natural without introducing variable names

Last class: English to Predicate Logic (Domain Restriction)

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"All red cats like tofu"

$$\forall x\ ((\mathrm{Red}(x) \wedge \mathrm{Cat}(x)) \rightarrow \mathrm{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\ ((\mathrm{Red}(y) \wedge \mathrm{Cat}(y)) \wedge \neg\mathrm{LikesTofu}(y))$$

Last class: Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall$x PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Domain of Discourse
{plum, apple}

(*) PurpleFruit(plum) $\wedge$ PurpleFruit(apple)

(a) PurpleFruit(plum) $\vee$ PurpleFruit(apple)

(b) $\neg$ PurpleFruit(plum) $\vee$ $\neg$ PurpleFruit(apple)

(c) $\neg$ PurpleFruit(plum) $\wedge$ $\neg$ PurpleFruit(apple)

Last class: De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

Intuition: $\forall$ is like a giant AND over the domain

$\exists$ is like a giant OR over the domain

Last class: De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

These are equivalent but not equal

They have different English translations, e.g.:

There is no unicorn $\neg \exists x\, \text{Unicorn}(x)$

Every animal is not a unicorn $\forall x \neg \text{Unicorn}(x)$

Last class: De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

"There is no integer at least as large as every other integer"

$$\neg \exists x\, \forall y\, (x \geq y)$$
$$\equiv \forall x\, \neg \forall y\, (x \geq y)$$
$$\equiv \forall x\, \exists y\, \neg (x \geq y)$$
$$\equiv \forall x\, \exists y\, (y > x)$$

"For every integer, there is a larger integer"

De Morgan’s Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

“No even prime is greater than 2”

$$\neg \exists x\, (\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$$
$$\equiv \forall x\, \neg(\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$$
$$\equiv \forall x\, (\neg(\text{Even}(x) \wedge \text{Prime}(x)) \vee \neg\text{Greater}(x, 2))$$
$$\equiv \forall x\, ((\text{Even}(x) \wedge \text{Prime}(x)) \rightarrow \neg\text{Greater}(x, 2))$$
$$\equiv \forall x\, ((\text{Even}(x) \wedge \text{Prime}(x)) \rightarrow \text{LessEq}(x, 2))$$

“Every even prime is less than or equal to 2.”

De Morgan's Laws for Quantifiers

We just saw that

$$\neg \exists x\, (P(x) \wedge R(x)) \equiv \forall x\, (P(x) \rightarrow \neg R(x))$$

Can similarly show that

$$\neg \forall x\, (P(x) \rightarrow R(x)) \equiv \exists x\, (P(x) \wedge \neg R(x))$$

De Morgan's Laws respect domain restrictions!
(It leaves them in place and only negates the other parts.)

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Remain true when domain restrictions are used:

$$\neg \exists x\, (P(x) \wedge R(x)) \equiv \forall x\, (P(x) \rightarrow \neg R(x))$$
$$\neg \forall x\, (P(x) \rightarrow R(x)) \equiv \exists x\, (P(x) \wedge \neg R(x))$$

Scope of Quantifiers

$\exists x\ (P(x) \land Q(x))$ **vs.** $(\exists x\ P(x)) \land (\exists x\ Q(x))$

Scope of Quantifiers

$\exists x\ (P(x) \wedge Q(x))$ **vs.** $(\exists x\ P(x)) \wedge (\exists x\ Q(x))$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists y$ Greater (y, x)

$\equiv \exists z$ Greater (z, x)

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\forall x\, (\exists y\, (P(x,y) \rightarrow \forall x\, Q(y, x)))$$

Quantifier "Style"

$$\forall x(\exists y\,(P(x,y) \rightarrow \forall\, x\, Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- Quantified variable names don't matter

$$\forall x\, \exists y\, P(x, y) \equiv \forall a\, \exists b\, P(a, b)$$

- Positions of quantifiers can sometimes change

$$\forall x\, (Q(x) \wedge \exists y\, P(x, y)) \equiv \forall x\, \exists y\, (Q(x) \wedge P(x, y))$$

- But: order is important...

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

"There is a number greater than or equal to all numbers."

$\exists x \; \forall y \; GreaterEq(x, y)))$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

Domain of Discourse

{1, 2, 3, 4}

Predicate Definitions

GreaterEq(x, y) ::= "$x \geq y$"

"There is a number greater than or equal to all numbers."

$\exists x \, \forall y$ GreaterEq(x, y)))

"Every number has a number greater than or equal to it."

$\forall y \, \exists x$ GreaterEq(x, y)))

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

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GreaterEq(x, y) ::= "x ≥ y"

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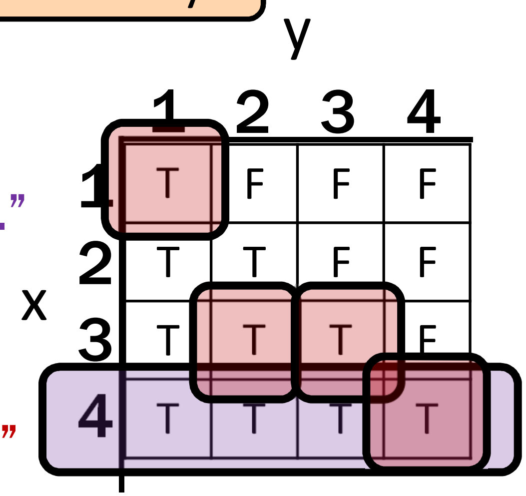

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Important: both include the case x = y

Different names does not imply different objects!

Quantification with Two Variables

	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

expression	when **true**	when **false**
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

p	*q*	A	B
T	T	T	
T	F	T	
F	T	F	
F	F	F	

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

p	*q*	A	B
T	T	T	T
T	F	T	T
F	T	F	
F	F	F	

Given that A is true, we see that B is also true.

$$A \Rightarrow B$$

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

p	*q*	A	B
T	T	T	T
T	F	T	T
F	T	F	?
F	F	F	?

When we zoom out, what have we proven?

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where B is true:

p	q	A	B	$A \rightarrow B$
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

When we zoom out, what have we proven?

$$(A \rightarrow B) \equiv \mathbf{T}$$

New Perspective

Equivalences

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ are the same

Inference

$A \Rightarrow B$ and $(A \rightarrow B) \equiv T$ are the same

Can do the inference by zooming in
to the rows where A is true

Applications of Logical Inference

- Software Engineering
  – Express desired properties of program as set of logical constraints
  – Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  – Automated reasoning
- Algorithm design and analysis
  – e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  – Express desired outcome as set of constraints
  – Automatically apply logic inference to derive solution

Proofs

- Start with given facts (hypotheses)
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If A and A → B are both true, then B must be true
- Write this rule as

$$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$

- Given:
  - If it is Friday, then you have a 311 class today.
  - It is Friday.
- Therefore, by Modus Ponens:
  - You have a 311 class today.

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.		
5.		

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

My First Proof!

Show that r follows from p, p → q, and q → r

1.	p	Given
2.	$p \to q$	Given
3.	$q \to r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Modus Ponens $\dfrac{A\ ;\ A \to B}{\therefore\ B}$

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Modus Ponens $\dfrac{A \,;\, A \rightarrow B}{\therefore\ B}$

Inference Rules

If A is true and B is true ....

Requirements: $\underline{\;A \;;\; B\;}$

Conclusions: $\therefore\ C \;,\; D$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \;;\; A \rightarrow B}{\therefore\ \quad B}$$

If I have A and $A \rightarrow B$ both true,
Then B must be true.

Axioms: Special inference rules

If I have nothing...

Requirements:

Conclusions: ∴ C , D

Then, C must be true

Then D must be true

Example (Excluded Middle):

$$\therefore\ A \vee \neg A$$

$A \vee \neg A$ must be true.

Simple Propositional Inference Rules

Two inference rules per binary connective,
one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \land B}{\therefore A,\ B}$$

Intro ∧

$$\frac{A\ ;\ B}{\therefore A \land B}$$

Elim ∨

$$\frac{A \lor B\ ;\ \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \lor B,\ B \lor A}$$

Modus Ponens

$$\frac{A\ ;\ A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

Proofs

Show that r follows from p, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{A \;;\; A \rightarrow B}{\therefore\; B}$$

$$\frac{A \wedge B}{\therefore\; A,\, B}$$

$$\frac{A \;;\; B}{\therefore\; A \wedge B}$$

Proofs

Show that r follows from $p, p \rightarrow q,$ and $p \wedge q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	q	MP: 1, 2
4.	$p \wedge q$	Intro $\wedge$: 1, 3
5.	$p \wedge q \rightarrow r$	Given
6.	r	MP: 4, 5

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

p ; $p \rightarrow q$
MP
p ; q
Intro $\wedge$
$p \wedge q$; $p \wedge q \rightarrow r$
MP
r

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

First: Write down givens and goal

20. $\neg r$?

Idea: Work backwards!

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

20. $\neg r$ MP: 2, ?

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim $\rightarrow$" which is MP.

Proofs

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove q...
  - Notice that at this point, if we prove q, we've proven $\neg r$...

Proofs

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

1.	$p \wedge s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

This looks like or-elimination.

Elim ∨ $\dfrac{A \vee B \,;\, \neg A}{\therefore B}$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

18. $\neg\neg s$?

$\neg\neg s$ doesn't show up in the givens but s does and we can use equivalences

19. q $\vee$ Elim: 3, 18
20. $\neg r$ MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	?
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	$\wedge$ Elim: 1
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

No holes left! We just need to clean up a bit.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \vee q$	Given
4.	s	$\wedge$ Elim: 1
5.	$\neg\neg s$	Double Negation: 4
6.	q	$\vee$ Elim: 3, 5
7.	$\neg r$	MP: 2, 6

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

  e.g. $(p \to r) \vee q \equiv (\neg p \vee r) \vee q$

- Inference rules only can be applied to whole formulas (not correct otherwise).

  e.g. 1. $p \to r$ given

  ~~2. $(p \vee q) \to r$ intro $\vee$ from 1.~~

  Does not follow! e.g . p=F, q=T, r=F

To Prove An Implication: $A \rightarrow B$

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is a proof that "Given A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \rightarrow B$ is true

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Example: Prove $p \rightarrow (p \vee q)$.

proof subroutine

Indent proof subroutine $\Rightarrow$

1.1.	p	Assumption
1.2.	$p \vee q$	Intro $\vee$: 1
1.	$p \rightarrow (p \vee q)$	Direct Proof

Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from q and $(p \wedge q) \rightarrow r$

1.	q	Given
2.	$(p \wedge q) \rightarrow r$	Given
3.1.	p	Assumption
3.2.	$p \wedge q$	Intro $\wedge$: 1, 3.1
3.3.	r	MP: 2, 3.2
3.	$p \rightarrow r$	Direct Proof

This is a proof of $p \rightarrow r$

If we know p is true… Then, we've shown r is true

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens…

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1.	$p \wedge q$	Assumption
1.2.	p	Elim $\wedge$: 1.1
1.3.	$p \vee q$	Intro $\vee$: 1.2
1.	$(p \wedge q) \rightarrow (p \vee q)$	Direct Proof

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.