Section 04: Propositions and Proofs

1. Formal Spoofs

For each of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $\exists z\ \forall x\ P(x, z)$ follows from $\forall x\ \exists y\ P(x, y)$.

1.	$\forall x\ \exists y\ P(x, y)$	[Given]
2.	$\forall x\ P(x, c)$	[$\exists$ Elim: 1, c special]
3.	$\exists z\ \forall x P(x, z)$	[$\exists$ Intro: 2]

(b) Show that $\exists z\ (P(z) \land Q(z))$ follows from $\forall x\ P(x)$ and $\exists y\ Q(y)$.

1.	$\forall x\ P(x)$	[Given]
2.	$\exists y\ Q(y)$	[Given]
3.	Let z be arbitrary	
4.	$P(z)$	[$\forall$ Elim: 1]
5.	$Q(z)$	[$\exists$ Elim: 2, let z be the object that satisfies $Q(z)$]
6.	$P(z) \land Q(z)$	[$\land$ Intro: 4, 5]
7.	$\exists z\ P(z) \land Q(z)$	[$\exists$ Intro: 6]

2. Just The Setup

For each of these statements,

- Translate the sentence into predicate logic.
- Write the first few and last few steps of an inference proof of the statement (you do not need to write the middle – just enough to introduce all givens and assumptions and the conclusion at the end)
- Write the first few sentences and last few sentences of the English proof.

(a) The product of an even integer and an odd integer is even.

(b) There is an integer x s.t. $x^2 > 10$ and $3x$ is even.

(c) For every integer n, there is a prime number p greater than n.

(d) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ for any sets A, B, C.

3. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\{\},\ \{\{\}\},\ \{\{\}, \{\}\},\ \{\{\}, \{\}, \{\}\},\ \dots\}$

(c) $C = A \times (B \cup \{7\})$

(d) $D = \varnothing$

(e) $E = \{\varnothing\}$

(f) $F = \mathcal{P}(\{\varnothing\})$

4. Set = Set

Prove the following set identities. Write both a formal inference proof **and** an English proof.

(a) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \backslash B$ for any sets A, B.

(b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

5. Set Equality

(a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

(b) Let $\mathcal{U}$ be the universal set. Show that $\overline{\overline{X}} = X$.

6. Trickier Set Theory

Note, this problem requires a little more thinking. The solution will cover both the answer as well as the intuition used to arrive at it.

Show that for any set X and any set A such that $A \in \mathcal{P}(X)$, there exists a set $B \in \mathcal{P}(X)$ such that $A \cap B = \emptyset$ and $A \cup B = X$.