Relations,Graphs
Midterm Misconceptions

Properties of relations

What do we do with relations? Usually we prove properties about them.

Symmetry

A binary relation R on a set S is "symmetric" iff
for all $a, b \in S$, $[(a, b) \in R \rightarrow (\mathrm{b}, \mathrm{a}) \in R]$

$=$ on Σ^* is symmetric, for all $a, b \in \Sigma^*$ if $a = b$ then $b = a$.
$\subseteq$ is not symmetric on $\mathcal{P}(\mathcal{U})$ – $\{1,2,3\} \subseteq \{1,2,3,4\}$ but $\{1,2,3,4\} \nsubseteq \{1,2,3\}$

Transitivity

A binary relation R on a set S is "transitive" iff
for all $a, b, c \in S$, $[(a, b) \in R \wedge (b, c) \in R \rightarrow (\mathrm{a}, \mathrm{c}) \in R]$

$=$ on Σ^* is transitive, for all $a, b, c \in \Sigma^*$ if $a = b$ and $b = \mathrm{c}$ then $a = c$.
$\subseteq$ is transitive on $\mathcal{P}(\mathcal{U})$ – for any sets A, B, C if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
$\in$ is not a transitive relation – $1 \in \{1,2,3\}$, $\{1,2,3\} \in \mathcal{P}(\{1,2,3\})$ but $1 \notin \mathcal{P}(\{1,2,3\})$.

Warm up

Show that $a \equiv b \ (mod\ n)$ if and only if $b \equiv a (mod\ n)$

$a \equiv b(mod\ n) \leftrightarrow n|(b-a) \leftrightarrow nk = b - a(\text{for } k \in \mathbb{Z}) \leftrightarrow$

$n(-k) = a - b(\text{for } -\mathrm{k} \in \mathbb{Z}) \leftrightarrow n|(a-b) \leftrightarrow b \equiv a(mod\ n)$

This was a proof that the relation $\{(a, b) : a \equiv b(mod\ n)\}$ is symmetric!

It was actually overkill to show if and only if. Showing just one direction turns out to be enough!

this is the form of the division theorem for $(a-n)\%n$. Since the division theorem guarantees a unique integer, $(a-n)\%n = (a\%n)$

You've also done a proof of transitivity!

5. Divide[s] we fall [14 points]

(a) Write an English proof showing that for any **positive** integers p, q, r if $p \mid q$ and $q \mid r$ then $p \mid r$. [8 points]

You did this proof on HW4. You were showing:
$\mid$ is a transitive relation on $\mathbb{Z}^+$.

More Properties of relations

What do we do with relations? Usually we prove properties about them.

Antisymmetry
A binary relation R on a set S is "antisymmetric" iff
for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \rightarrow (\mathbf{b}, \mathbf{a}) \notin R]$

$\leq$ is antisymmetric on $\mathbb{Z}$

Reflexivity
A binary relation R on a set S is "reflexive" iff
for all $a \in S$, $[(a, a) \in R]$

$\leq$ is reflexive on $\mathbb{Z}$

$\leq$

You've proven antisymmetry too!

(b) Write an English proof showing that for any **positive** integers p, q if $p \mid q$ and $q \mid p$, then $p = q$. For this problem, you may not use the result of Section 4's problem 5a as a fact, but you may find that proof useful to model yours after. [6 points]

Antisymmetry

A binary relation R on a set S is "antisymmetric" iff for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \rightarrow (\mathrm{b}, \mathrm{a}) \notin R]$

You showed | is antisymmetric on $\mathbb{Z}^+$

for all $a, b \in S$, $[(a, b) \in R \wedge (\mathrm{b}, \mathrm{a}) \in R \rightarrow a = b]$ is equivalent to the definition in the box above

The box version is easier to understand, the other version is usually easier to prove.

Try a few of your own

Fill out the poll everywhere for Activity Credit!

Go to pollev.com/cse311 and login with your UW identity
Or text cse311 to 37607

Decide whether each of these relations are

Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$ on $\mathcal{P}(\mathcal{U})$

$\geq$ on $\mathbb{Z}$

$>$ on $\mathbb{R}$

$|$ on $\mathbb{Z}^+$

$|$ on $\mathbb{Z}$

$\equiv (mod\ 3)$ on $\mathbb{Z}$

Symmetry: for all $a, b \in S$, $[(a, b) \in R \rightarrow (\mathrm{b}, \mathrm{a}) \in R]$

Antisymmetry: for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \rightarrow (\mathrm{b}, \mathrm{a}) \notin R]$

Transitivity: for all $a, b, c \in S$, $[(a, b) \in R \wedge (b, c) \in R \rightarrow (\mathrm{a}, \mathrm{c}) \in R]$

Reflexivity: for all $a \in S$, $[(a, a) \in R]$

Try a few of your own

Symmetry: for all $a, b \in S$, $[(a, b) \in R \rightarrow (\mathrm{b}, \mathrm{a}) \in R]$

Antisymmetry: for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \rightarrow (\mathrm{b}, \mathrm{a}) \notin R]$

Transitivity: for all $a, b, c \in S$, $[(a, b) \in R \wedge (b, c) \in R \rightarrow (\mathrm{a}, \mathrm{c}) \in R]$

Reflexivity: for all $a \in S$, $[(a, a) \in R]$

Decide whether each of these relations are

Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$ on $\mathcal{P}(\mathcal{U})$ reflexive, antisymmetric, transitive

$\geq$ on $\mathbb{Z}$ reflexive, antisymmetric, transitive

$>$ on $\mathbb{R}$ antisymmetric, transitive

$|$ on $\mathbb{Z}^+$ reflexive, antisymmetric, transitive

$|$ on $\mathbb{Z}$ reflexive, transitive

$\equiv (mod\ 3)$ on $\mathbb{Z}$ reflexive, symmetric, transitive

Two Prototype Relations

A lot of fundamental relations follow one of two prototypes:

Equivalence Relation

A relation that is reflexive, symmetric, and transitive is called an "equivalence relation"

Partial Order Relation

A relation that is reflexive, antisymmetric, and transitive is called a "partial order"

Equivalence Relations

Equivalence relations "act kinda like equals"

$\equiv \pmod{n}$ is an equivalence relation.

$\equiv$ on compound propositions is an equivalence relation.

Fun fact: Equivalence relations "partition" their elements.

An equivalence relation R on S divides S into sets $S_1, \ldots S_k$ such that.

$\forall s\ (s \in S_i \text{ for some } i)$

$\forall s, s'$ $(s, s' \in S_i$ for some i if and only if $(s, s') \in R)$

$S_i \cap S_j = \emptyset$ for all $i \neq j$

Partial Orders

Partial Orders “behave kinda like less than or equal to”

In the sense that they put things in order

But it’s only kinda like less than – it’s possible that some elements can’t be compared.

$|$ on $\mathbb{Z}^+$ is a partial order

$\subseteq$ on $\mathcal{P}(\mathcal{U})$ is a partial order

x is a prerequisite of (or-equal-to) y is a partial order on CSE courses

Why Bother?

If you prove facts about all equivalence relations or all partial orders, you instantly get facts in lots of different contexts.

If you learn to recognize partial orders or equivalence relations, you can get **a lot** of intuition for new concepts in a short amount of time.

Combining Relations

Given a relation R from A to B

And a relation S from B to C,

The relation $S \circ R$ from A to C is

$\{(a, c) : \exists b[(a, b) \in R \land (b, c) \in S]\}$

Yes, I promise it's $S \circ R$ not $R \circ S$ – it makes more sense if you think about relations $(x, f(x))$ and $(x, g(x))$

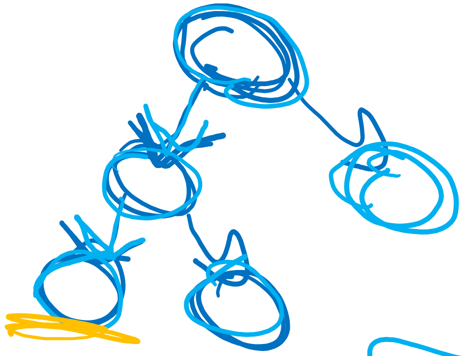
But also don't spend a ton of energy worrying about the order, we almost always care about $R \circ R$, where order doesn't matter.

Combining Relations

To combine relations, it’s a lot easier if we can see what’s happening.

We’ll use a representation of a directed graph

Directed Graphs



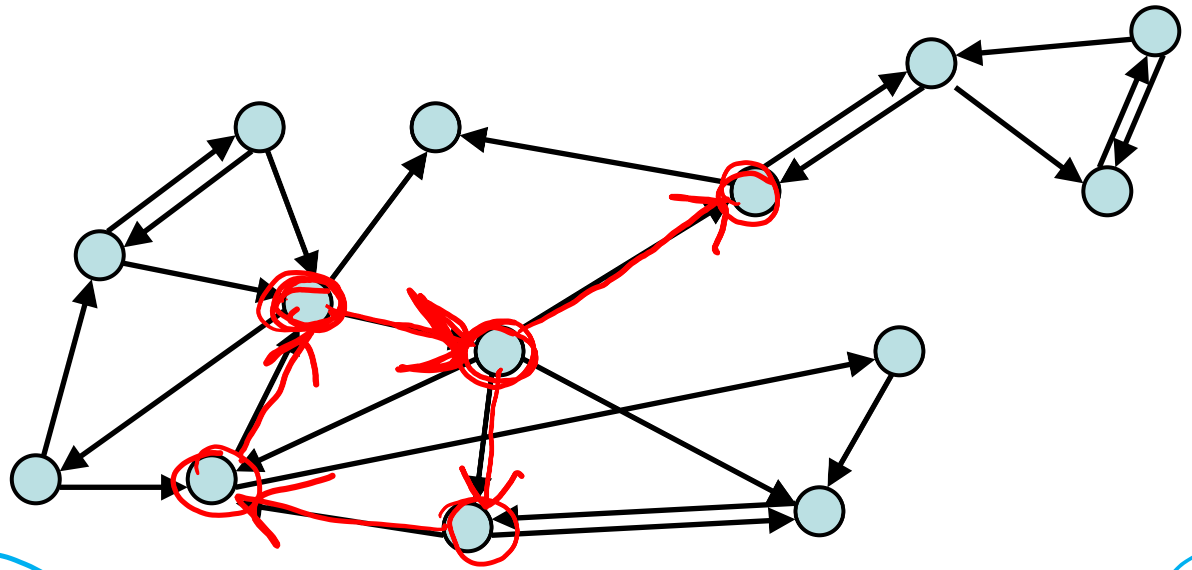
$G = (V, E)$

V is a set of vertices (an underlying set of elements)

E is a set of edges (ordered pairs of vertices; i.e. connections from one to the next).

(parent, child)

Path $v_0, v_1, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$
Simple Path: path with all v_i distinct
Cycle: path with $v_0 = v_k$ (and $k > 0$)
Simple Cycle: simple path plus edge (v_k, v_0) with $k > 0$



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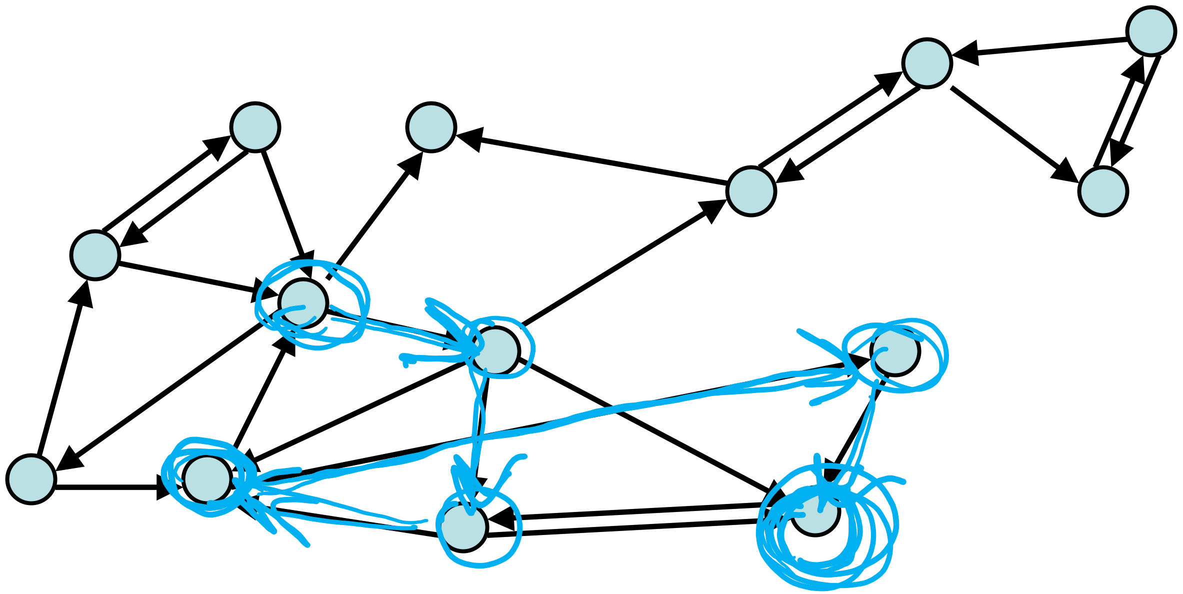

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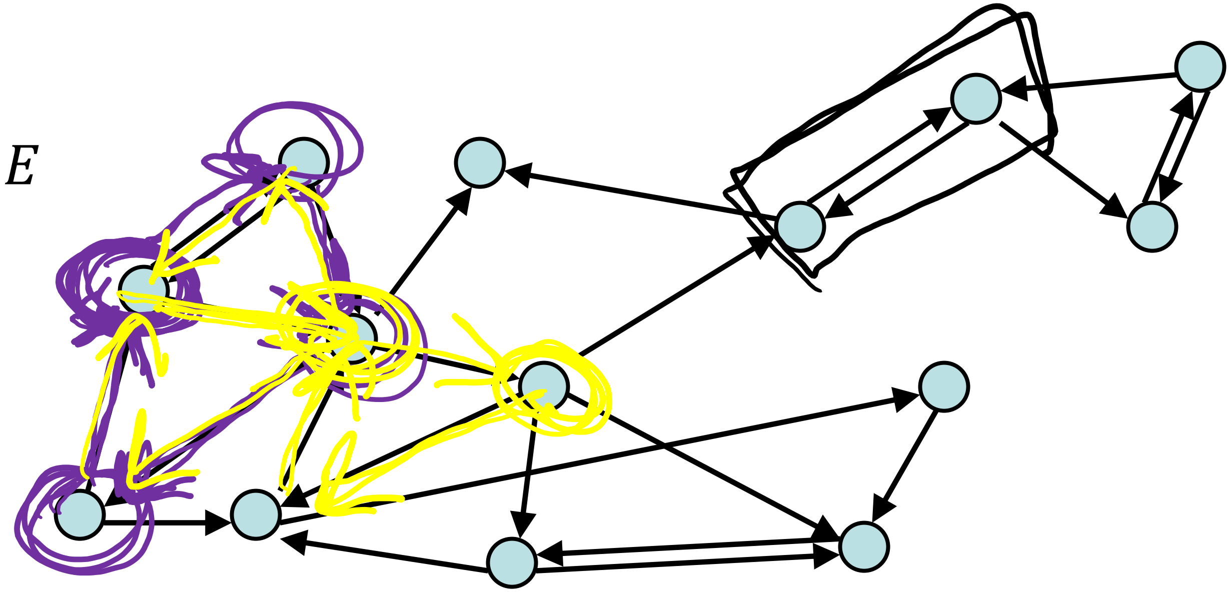

Directed Graphs

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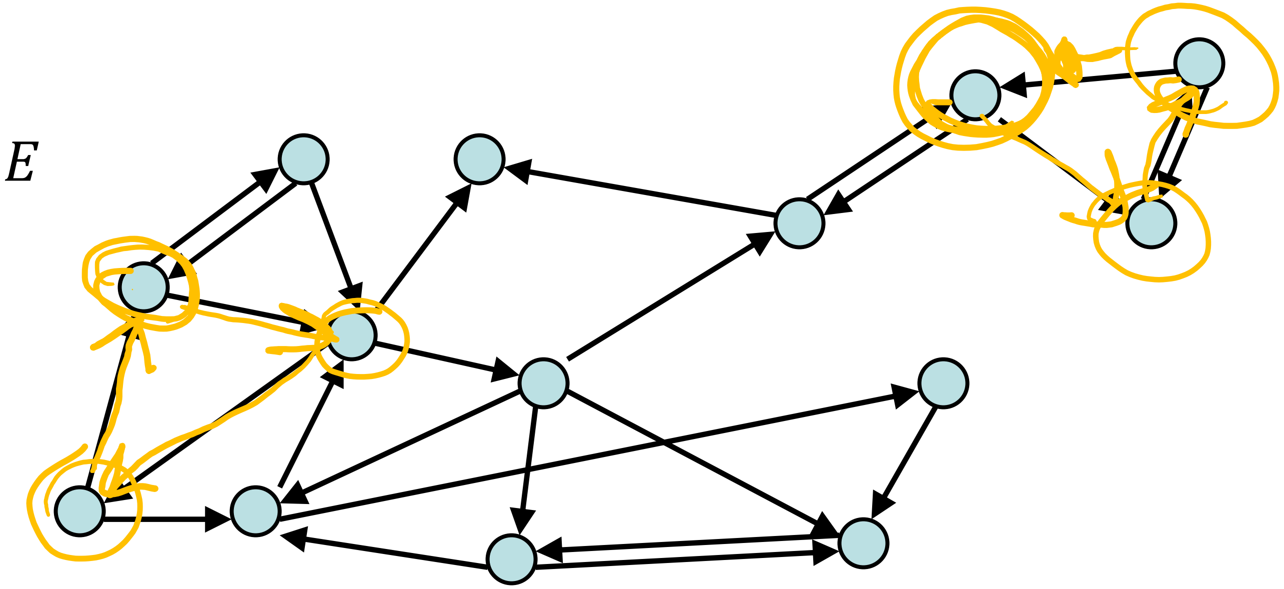
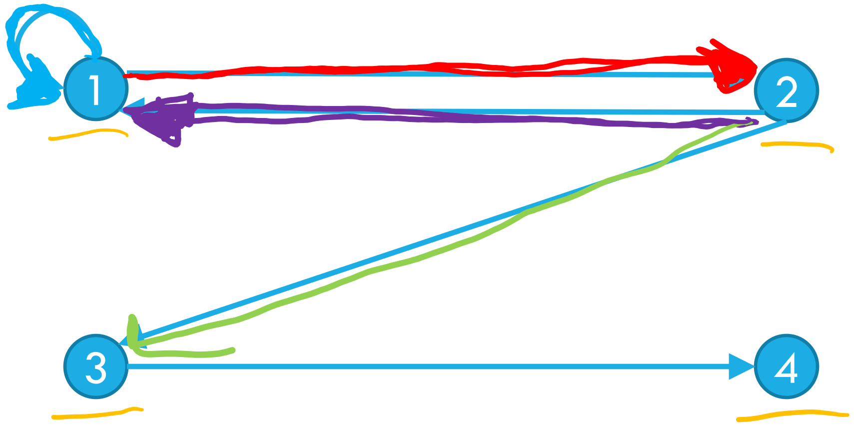
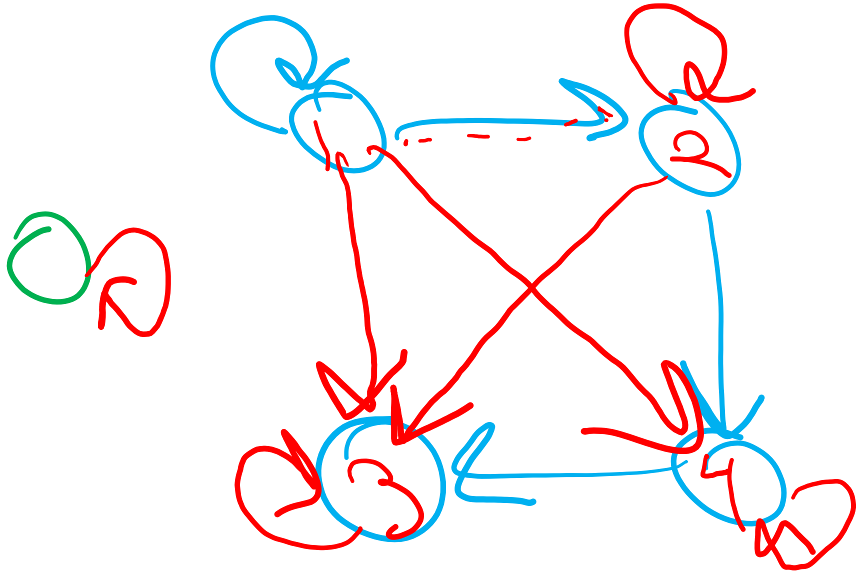


Representing Relations

To represent a relation R on a set A, have a vertex for each element of A and have an edge (a, b) for every pair in R.

Let A be $\{1,2,3,4\}$ and R be $\{(1,1),(1,2),(2,1),(2,3),(3,4)\}$

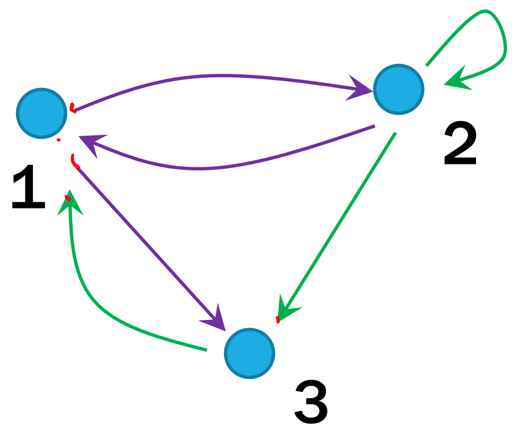


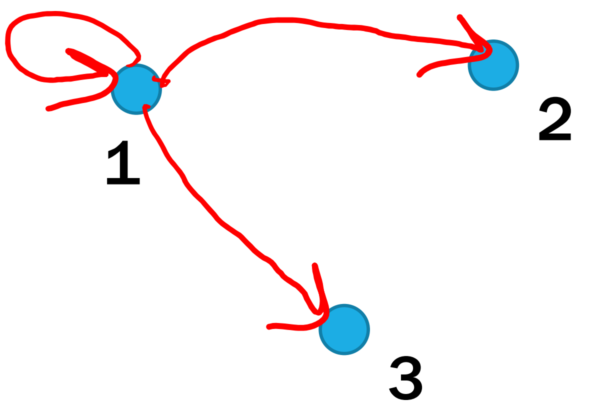


Combining Relations

If $\boldsymbol{S} = \{(\mathbf{2},\mathbf{2}), (\mathbf{2},\mathbf{3}), (\mathbf{3},\mathbf{1})\}$ and $\boldsymbol{R} = \{(\mathbf{1},\mathbf{2}), (\mathbf{2},\mathbf{1}), (\mathbf{1},\mathbf{3})\}$

Compute $\boldsymbol{S} \circ \boldsymbol{R}$ i.e. every pair (a, c) with a b with $(a, b) \in R$ and $(b, c) \in S$





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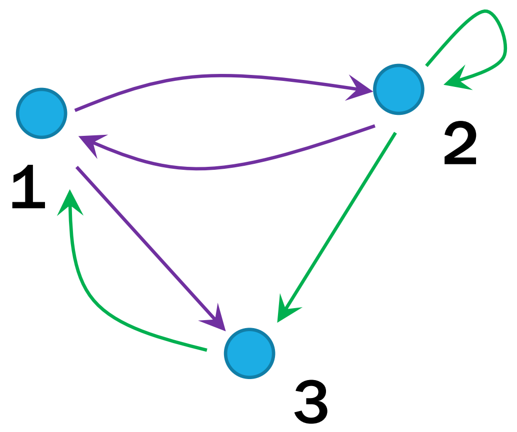
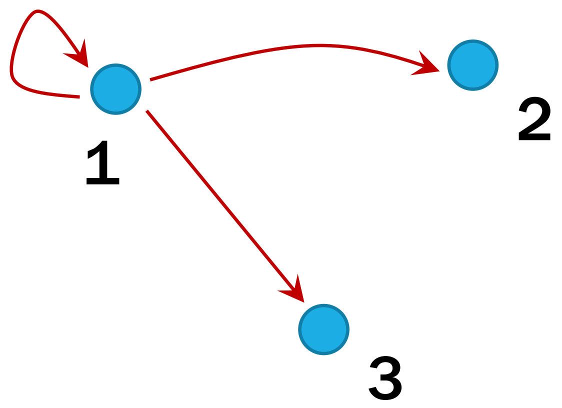
Compute $\boldsymbol{S} \circ \boldsymbol{R}$ i.e. every pair (a, c) with a b with $(a, b) \in R$ and $(b, c) \in S$





Combining Relations

Let R be a relation on A.

Define R^0 as $\{(a, a) : a \in A\}$

$R^k = R^{k-1} \circ R$

$(a, b) \in R^k$ if and only if there is a path of length k from a to b in R.

We can find that on the graph!

More Powers of R.

For two vertices in a graph, a can reach b if there is a path from a to b.

Let R be a relation on the set A. The connectivity relation R^* consists of all pairs (a, b) such that a can reach b (i.e. there is a path from a to b in R)

$R^* = \bigcup_{k=0}^{\infty} R^k$

Note we're starting from 0 (the textbook makes the unusual choice of starting from $k = 1$).

What's the point of R^*

R^* is also the "reflexive-transitive closure of R."

It answers the question "what's the minimum amount of edges I would need to add to R to make it reflexive and transitive.

Why care about that? The transitive-reflexive closure can be a summary of data – you might want to precompute it so you can easily check if a can reach b instead of recomputing it every time.

Calculating R^*

For every vertex, add an edge from it to itself

While there is an edge (a, b) and an edge (b, c) but not the edge (a, c), add (a, c).

How would you do this in code?

You could just iteratively add edges (would take about $O(n^3)$ time if you have n elements in the set).

But there are tricks to do it faster (by about an n factor) – take CSE 421 to learn them!

Relations and Graphs

Describe how each property will show up in the graph of a relation.

Reflexive

Symmetric

Antisymmetric

Transitive

Relations and Graphs

Describe how each property will show up in the graph of a relation.

Reflexive

Every vertex has a “self-loop” (an edge from the vertex to itself)

Symmetric

Every edge has its “reverse edge” (going the other way) also in the graph.

Antisymmetric

No edge has its “reverse edge” (going the other way) also in the graph.

Transitive

If there’s a length-2 path from a to b then there’s a direct edge from a to b

Midterm Misconceptions

Induction Problem

Call a line "properly ordered" if it meets all of those requirements.
Show that for all $n \geq 2$, in every properly ordered line there are two people wearing gold hats next to each other.

$P(n)$: "Every properly ordered line with n pairs has two consecutive people wearing gold hats."

Base Case: $n = 2$ P G G P

We need to show a $\forall$ statement in the inductive step.

To prove a for all statement, the first thing we do in our proof is…

Introduce an arbitrary variable!

Induction Problem

So if you didn't start with "let L be an arbitrary properly ordered line with $k+1$ pairs of people" you didn't start in the right place.

If you started with "an arbitrary properly ordered line with k pairs"

There's not formally a way to argue that "by listing out all the possible alterations I could think of, I'll end up with all the possible lines of length $k+1$"

You might have (you probably did) but it's still not a rigorous argument of a forall statement if you don't start with an arbitrary line of length $k+1$.

Induction Problem

This kind of attempted induction argument (where you "build up" to a supposedly arbitrary element from an arbitrary smaller element) easily hides bugs. For that reason it's not logically valid.

See: HW6 P6.

Never ever ever try to prove a "for all" induction by building up (ever).

Always start with the arbitrary big thing (the $k+1$ thing) and find the smaller thing inside.

There is no rule of inference that says "I started with an arbitrary thing and did some alterations and it's now an arbitrary other thing"

Induction Problem

But wait…don’t we just do that when we prove inequalities by induction?

Nope!

1. Inequalities aren’t for-all statements (or if they are you introduce the variable at the start, like we did for that string induction proof)

2. We prove inequalities the normal way we prove inequalities (either starting from a fact you know and deriving the desired inequality, or starting from the left hand side and altering it until you get the right hand side).

Induction Problem

But wait, don't we "build up" when we do structural induction?

Nope!

Basis:

Recursive:

Exclusion: nothing else is in S

The recursive definition in structural induction guarantees us what the arbitrary element looks like…it's made up of two 'smaller' elements in the set.

…and the template just lets us skip the words "let T be arbitrary, by the recursive definition, T is of the form…"

Induction Problem

But wait, that stamp collecting problem. We definitely started with the small one there.

The stamp collecting induction was an exists statement (there is a way to build $k + 1$). So yeah, we definitely didn't have anything arbitrary there.

Nor would we expect to – it was an exists statement!

Set Problems

Notes from the TAs

Be careful with set-builder notation

Using variables you've defined in spots where dummy variables are expected:

1. Does not mean what you think it means.
2. "Hurts [your TA's] brain"

Dummy Variables

A lot like a local variable in Java.

It means something only inside its method.

$\int 5x^2dx$ x is a dummy variable. It means something inside the integral, (so you can write dx) but wouldn't necessarily mean anything outside.

$\exists x(P(x) \wedge Q(x))$ x is a dummy variable.

$\{y : y^2 \geq 5\}$ y is a dummy variable

Dummy Variables

So if you said something like
Let y be an arbitrary element of output,

Consider $\{y : y = x\}$ this y is not that y

Set Proofs

If you're showing $A \subseteq B$

Your first step should **always** be

Let x be an arbitrary element of A.

A lot of you had attempted proofs where you tried to write $output(f, A \cap B) = \{y: \exists x\ f(x) = y\}$ and modify the inside

$\{y : \exists x\ \ldots\}$

Don't do this. It's never how you do a set proof.

Set problems

This was a hard problem.

It's what we call a "synthesis" problem – applying familiar ideas and techniques in new combinations.

You should expect these types of problems in all of your future courses if you haven't seen them already; the end goal of university education is synthesis.

You needed to combine set proofs, set builder notation, quantifier notation (both exists and for-all) to do this problem.

When a problem is hard, it's easy to get overwhelmed.

Take a deep breath, and do the 4-step process.

1. What do the words in the statement mean?
2. What does the statement as a whole mean?
3. Where do I start?
4. Where is my target?

"What types are these objects?"

For any function f and any set S, define output$(f, S) = \{y : \exists x(x \in S \land f(x) = y)\}$

For example, output$(x^2, \{1, 2, 3\}) = \{y : \exists x(x \in \{1, 2, 3\} \land f(x) = y)\} = \{1, 4, 9\}$, and output$(x^2, \{-1, 1\}) = \{1\}$.

(a) Show that output$(f, A \cap B) \subseteq$ output$(f, A) \cap$ output(f, B). [10 points]

Let f be an arbitrary function, let A, B be arbitrary sets.

Let y be an arbitrary element of output$(f, A \cap B)$

Therefore $y \in$ output$(f, A) \cap$ output(f, B).

For any function f and any set S, define output$(f, S) = \{y : \exists x(x \in S \wedge f(x) = y)\}$

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Let f be an arbitrary function, let A, B be arbitrary sets.

Let y be an arbitrary element of output$(f, A \cap B)$

By defn of output, there is an $x \in A \cap B$ s.t. $f(x) = y$

By defn of intersection $x \in A$ and $x \in B$

Since $x \in A$ and $f(x) = y$, $y \in$ output(f, A)

" $x \in B$ " " $y \in$ output(f, B)

Then

$y \in$ output$(f, A) \wedge y \in$ output(f, B)

So $y \in$ output$(f, A) \cap$ output(f, B)

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Let y be an arbitrary element of output$(f, A \cap B)$

By definition of output, there is an x such that $x \in A \cap B$ and $f(x) = y$

y ∈output(f, A) and y ∈output(f, B)

So y ∈output(f, A) ∩output(f, B)

For any function f and any set S, define output$(f, S) = \{y : \exists x(x \in S \wedge f(x) = y)\}$

For example, output$(x^2, \{1, 2, 3\}) = \{y : \exists x(x \in \{1, 2, 3\} \wedge f(x) = y)\} = \{1, 4, 9\}$, and output$(x^2, \{-1, 1\}) = \{1\}$.

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Let f be an arbitrary function, let A, B be arbitrary sets.

Let y be an arbitrary element of output$(f, A \cap B)$

By definition of output, there is an x such that $x \in A \cap B$ and $f(x) = y$

Since $x \in A \cap B$ $x \in A$ and $x \in B$.

So by definition of output, $f(x) = y \in$output(f, A) and $f(x) = y \in$output(f, B)

$y \in$output(f, A) and $y \in$output(f, B)

So $y \in$output$(f, A) \cap$output(f, B)