Section 07: Solutions

1. Structural Induction

(a) Consider the following recursive definition of strings.

Basis Step: "" is a string

Recursive Step: If X is a string and c is a character then $\mathsf{append}(c, X)$ is a string.

Recall the following recursive definition of the function len:

$$\begin{aligned}
\mathsf{len}("") &= 0 \\
\mathsf{len}(\mathsf{append}(c, X)) &= 1 + \mathsf{len}(X)
\end{aligned}$$

Now, consider the following recursive definition:

$$\begin{aligned}
\mathsf{double}("") &= "" \\
\mathsf{double}(\mathsf{append}(c, X)) &= \mathsf{append}(c, \mathsf{append}(c, \mathsf{double}(X))).
\end{aligned}$$

Prove that for any string X, $\mathsf{len}(\mathsf{double}(X)) = 2\mathsf{len}(X)$.

Solution:

> For a string X, let $\mathsf{P}(X)$ be "$\mathsf{len}(\mathsf{double}(X)) = 2\mathsf{len}(X)$". We prove $\mathsf{P}(X)$ for all strings X by structural induction on X.
>
> **Base Case (X = ""):** By definition, $\mathsf{len}(\mathsf{double}("")) = \mathsf{len}("") = 0 = 2 \cdot 0 = 2\mathsf{len}("")$, so $\mathsf{P}("")$ holds
>
> **Inductive Hypothesis:** Suppose $\mathsf{P}(X)$ holds for some arbitrary string X.
>
> **Inductive Step:** $\boxed{\text{Goal: Show that } \mathsf{P}(\mathsf{append}(c, X)) \text{ holds for any character } c.}$
>
> $$\begin{aligned}
\mathsf{len}(\mathsf{double}(\mathsf{append}(c, X))) &= \mathsf{len}(\mathsf{append}(c, \mathsf{append}(c, \mathsf{double}(X)))) && \text{[By Definition of } \mathsf{double}\text{]} \\
&= 1 + \mathsf{len}(\mathsf{append}(c, \mathsf{double}(X))) && \text{[By Definition of } \mathsf{len}\text{]} \\
&= 1 + 1 + \mathsf{len}(\mathsf{double}(X)) && \text{[By Definition of } \mathsf{len}\text{]} \\
&= 2 + 2\mathsf{len}(X) && \text{[By IH]} \\
&= 2(1 + \mathsf{len}(X)) && \text{[Algebra]} \\
&= 2(\mathsf{len}(\mathsf{append}(c, X))) && \text{[By Definition of } \mathsf{len}\text{]}
\end{aligned}$$
>
> This proves $\mathsf{P}(\mathsf{append}(c, X))$.
>
> **Conclusion:** $\mathsf{P}(X)$ holds for all strings X by structural induction.

(b) Consider the following definition of a (binary) **Tree**:

Basis Step: $\bullet$ is a **Tree**.

Recursive Step: If L is a **Tree** and R is a **Tree** then $\mathsf{Tree}(\bullet, L, R)$ is a **Tree**.

The function leaves returns the number of leaves of a **Tree**. It is defined as follows:

$$\begin{aligned}
\mathsf{leaves}(\bullet) &= 1 \\
\mathsf{leaves}(\mathsf{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R)
\end{aligned}$$

Also, recall the definition of size on trees:

$$\begin{aligned}\mathsf{size}(\bullet) &= 1\\ \mathsf{size}(\mathsf{Tree}(\bullet, L, R)) &= 1 + \mathsf{size}(L) + \mathsf{size}(R)\end{aligned}$$

Prove that $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$ for all Trees T.

Solution:

> For a tree T, let P be $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$. We prove P for all trees T by structural induction on T.
>
> **Base Case (T = •):** By definition of $\mathsf{leaves}(\bullet)$, $\mathsf{leaves}(\bullet) = 1$ and $\mathsf{size}(\bullet) = 1$. So, $\mathsf{leaves}(\bullet) = 1 \geq 1/2 + 1/2 = \mathsf{size}(\bullet)/2 + 1/2$, so $\mathsf{P}(\bullet)$ holds.
>
> **Inductive Hypothesis:** Suppose $\mathsf{P}(L)$ and $\mathsf{P}(R)$ hold for some arbitrary trees L, R.
>
> **Inductive Step:** ⎡Goal: Show that $\mathsf{P}(\mathsf{Tree}(\bullet, L, R))$ holds.⎤
>
> $$\begin{aligned}\mathsf{leaves}(\mathsf{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R) &&[\text{By Definition of } \mathsf{leaves}]\\ &\geq (\mathsf{size}(L)/2 + 1/2) + (\mathsf{size}(R)/2 + 1/2) &&[\text{By IH}]\\ &= (1/2 + \mathsf{size}(L)/2 + \mathsf{size}(R)/2) + 1/2 &&[\text{By Algebra}]\\ &= \frac{1 + \mathsf{size}(L) + \mathsf{size}(R)}{2} + 1/2 &&[\text{By Algebra}]\\ &= \mathsf{size}(T)/2 + 1/2 &&[\text{By Definition of } \mathsf{size}]\end{aligned}$$
>
> This proves $\mathsf{P}(\mathsf{Tree}(\bullet, L, R))$.
>
> **Conclusion:** Thus, $\mathsf{P}(T)$ holds for all trees T by structural induction.

(c) Prove the previous claim using strong induction. Define $P(n)$ as "all trees T of size n satisfy $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$". You may use the following facts:

- For any tree T we have $\mathsf{size}(T) \geq 1$.
- For any tree T, $\mathsf{size}(T) = 1$ if and only if $T = \bullet$.

If we wanted to prove these claims, we could do so by structural induction.

Note, in the inductive step you should start by letting T be an arbitrary tree of size $k + 1$.

Solution:

> Let $P(n)$ be "all trees T of size n satisfy $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$". We show $P(n)$ for all integers $n \geq 1$ by strong induction on n.
>
> **Base Case:** Let T be an arbitrary tree of size 1. The only tree with size 1 is •, so $T = \bullet$. By definition, $\mathsf{leaves}(T) = \mathsf{leaves}(\bullet) = 1$ and thus $\mathsf{size}(T) = 1 = 1/2 + 1/2 = \mathsf{size}(T)/2 + 1/2$. This shows the base case holds.
>
> **Inductive Hypothesis:** Suppose that $P(j)$ holds for all integers $j = 1, 2, \ldots, k$ for some arbitrary integer $k \geq 1$.
>
> **Inductive Step:** Let T be an arbitrary tree of size $k + 1$. Since $k + 1 > 1$, we must have $T \neq \bullet$. It follows from the definition of a tree that $T = \mathsf{Tree}(\bullet, L, R)$ for some trees L and R. By definition, we have $\mathsf{size}(T) = 1 + \mathsf{size}(L) + \mathsf{size}(R)$. Since sizes are non-negative, this equation shows $\mathsf{size}(T) > \mathsf{size}(L)$ and $\mathsf{size}(T) > \mathsf{size}(R)$ meaning we can apply the inductive hypothesis. This says that $\mathsf{leaves}(L) \geq \mathsf{size}(L)/2 + 1/2$ and $\mathsf{leaves}(R) \geq \mathsf{size}(R)/2 + 1/2$.

We have,

$$
\begin{aligned}
\mathsf{leaves}(T) &= \mathsf{leaves}(\mathtt{Tree}(\bullet, L, R)) \\
&= \mathsf{leaves}(L) + \mathsf{leaves}(R) && \text{[By Definition of \textsf{leaves}]} \\
&\geq (\mathsf{size}(L)/2 + 1/2) + (\mathsf{size}(R)/2 + 1/2) && \text{[By IH]} \\
&= (1/2 + \mathsf{size}(L)/2 + \mathsf{size}(R)/2) + 1/2 && \text{[By Algebra]} \\
&= \frac{1 + \mathsf{size}(L) + \mathsf{size}(R)}{2} + 1/2 && \text{[By Algebra]} \\
&= \mathsf{size}(T)/2 + 1/2 && \text{[By Definition of \textsf{size}]}
\end{aligned}
$$

This shows $P(k+1)$.

Conclusion: $P(n)$ holds for all integers $n \geq 1$ by the principle of strong induction.

Note, this proves the claim for all trees because every tree T has some size $s \geq 1$. Then $P(s)$ says that all trees of size s satisfy the claim, including T.

2. Regular Expressions

(a) Write a regular expression that matches base 10 numbers (e.g., there should be no leading zeroes).

Solution:

$$
0 \cup ((1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)^*)
$$

(b) Write a regular expression that matches all base-3 numbers that are divisible by 3.

Solution:

$$
0 \cup ((1 \cup 2)(0 \cup 1 \cup 2)^*0)
$$

(c) Write a regular expression that matches all binary strings that contain the substring "111", but not the substring "000".

Solution:

$$
(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)111(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)
$$

3. CFGs

(a) All binary strings that end in 00.

Solution:

$$
\mathbf{S} \to 0\mathbf{S} \mid 1\mathbf{S} \mid 00
$$

(b) All binary strings that contain at least three 1's.
   Solution:

> $$\mathbf{S} \to \mathbf{TTT}$$
> $$\mathbf{T} \to 0\mathbf{T} \mid \mathbf{T}0 \mid 1\mathbf{T} \mid 1$$

(c) All binary strings with an equal number of 1's and 0's.
   Solution:

> $$\mathbf{S} \to 0\mathbf{S}1\mathbf{S} \mid 1\mathbf{S}0\mathbf{S} \mid \varepsilon$$
>
> and
>
> $$\mathbf{S} \to \mathbf{SS} \mid 0\mathbf{S}1 \mid 1\mathbf{S}0 \mid \varepsilon$$
>
> both work. Note: The fact that all the strings generated have the property is easy to show (by induction) but the fact that one can generate all strings with the property is trickier. To argue this that each of these is grammars is enough one would need to consider how the difference between the # of 0's seen and the # of 1's seen occurs in prefixes of any string with the property.

4. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 3 or 7 at a time (every dog gets walked exactly once). Prove the dog walker can always split the n dogs into groups of 3 or 7.

Solution:

> Let $P(n)$ be "a group with n dogs can be split into groups of 3 or 7 dogs." We will prove $P(n)$ for all natural numbers $n \geq 12$ by strong induction.
>
> **Base Cases** $n = 12, 13, 14,$ **or** 15: $12 = 3 + 3 + 3 + 3$, $13 = 3 + 7 + 3$, $14 = 7 + 7$, So $P(12)$, $P(13)$, and $P(14)$ hold.
>
> **Inductive Hypothesis:** Assume that $P(12), \ldots, P(k)$ hold for some arbitrary $k \geq 14$.
>
> **Inductive Step:** | Goal: Show $k + 1$ dogs can be split into groups of size 3 or 7. |
>   We first form one group of 3 dogs. Then we can divide the remaining $k-2$ dogs into groups of 3 or 7 by the assumption $P(k-2)$. (Note that $k \geq 14$ and so $k-2 \geq 12$; thus, $P(k-2)$ is among our assumptions $P(12), \ldots, P(k)$.)
>
> **Conclusion:** $P(n)$ holds for all integers $n \geq 12$ by by principle of strong induction.

5. Reversing a Binary Tree

Consider the following definition of a (binary) **Tree**.

Basis Step Nil is a **Tree**.

Recursive Step If L is a **Tree**, R is a **Tree**, and x is an integer, then $\mathsf{Tree}(x, L, R)$ is a **Tree**.

The sum function returns the sum of all elements in a **Tree**.

$$\begin{aligned} \mathsf{sum}(\mathsf{Nil}) &= 0 \\ \mathsf{sum}(\mathsf{Tree}(x, L, R)) &= x + \mathsf{sum}(L) + \mathsf{sum}(R) \end{aligned}$$

The following recursively defined function produces the mirror image of a **Tree**.

$$\begin{aligned}
\mathsf{reverse}(\mathtt{Nil}) &= \mathtt{Nil} \\
\mathsf{reverse}(\mathtt{Tree}(x, L, R)) &= \mathtt{Tree}(x, \mathsf{reverse}(R), \mathsf{reverse}(L))
\end{aligned}$$

Show that, for all **Trees** T that

$$\mathsf{sum}(T) = \mathsf{sum}(\mathsf{reverse}(T))$$

Solution:

For a **Tree** T, let $P(T)$ be "$\mathsf{sum}(T) = \mathsf{sum}(\mathsf{reverse}(T))$". We show $P(T)$ for all **Trees** T by structural induction.

Base Case: By definition we have $\mathsf{reverse}(\mathtt{Nil}) = \mathtt{Nil}$. Applying sum to both sides we get $\mathsf{sum}(\mathtt{Nil}) = \mathsf{sum}(\mathsf{reverse}(\mathtt{Nil}))$, which is exactly $P(\mathtt{Nil})$, so the base case holds.

Inductive Hypothesis: Suppose $P(L)$ and $P(R)$ hold for some arbitrary **Trees** L and R.

Inductive Step: Let x be an arbitrary integer. $\boxed{\text{Goal: Show } P(\mathtt{Tree}(x, L, R)) \text{ holds.}}$

We have,

$$\begin{aligned}
\mathsf{sum}(\mathsf{reverse}(\mathtt{Tree}(x, L, R))) &= \mathsf{sum}(\mathtt{Tree}(x, \mathsf{reverse}(R), \mathsf{reverse}(L))) && \text{[Definition of } \mathsf{reverse}\text{]} \\
&= x + \mathsf{sum}(\mathsf{reverse}(R)) + \mathsf{sum}(\mathsf{reverse}(L)) && \text{[Definition of } \mathsf{sum}\text{]} \\
&= x + \mathsf{sum}(R) + \mathsf{sum}(L) && \text{[Inductive Hypothesis]} \\
&= x + \mathsf{sum}(L) + \mathsf{sum}(R) && \text{[Commutativity]} \\
&= \mathsf{sum}(\mathtt{Tree}(x, L, R)) && \text{[Definition of } \mathsf{sum}\text{]}
\end{aligned}$$

This shows $P(\mathtt{Tree}(x, L, R))$.

Conclusion: Therefore, $P(T)$ holds for all **Trees** T by structural induction.

6. Recursively Defined Sets of Strings

For each of the following, write a recursive definition of the sets satisfying the following properties. Briefly justify that your solution is correct.

(a) Binary strings of even length.

Solution:

Basis: $\varepsilon \in S$.
Recursive Step: If $x \in S$, then $x00, x01, x10, x11 \in S$.
Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

"Brief" Justification: We will show that $x \in S$ iff x has even length (i.e., $|x| = 2n$ for some $n \in \mathbb{N}$). (Note: "brief" is in quotes here. Try to write shorter explanations in your homework assignment when possible!)

Suppose $x \in S$. If x is the empty string, then it has length 0, which is even. Otherwise, x is built up from the empty string by repeated application of the recursive step, so it is of the form $x_1 x_2 ... x_n$, where each $x_i \in \{00, 01, 10, 11\}$. In that case, we can see that $|x| = |x_1| + |x_2| + \cdots + |x_n| = 2n$, which is even. Now, suppose that x has even length. If it's length is zero, then it is the empty string, which is in S. Otherwise, it has length 2n for some $n > 0$, and we can write x in the form $x_1 x_2 ... x_n$, where each $x_i \in \{00, 01, 10, 11\}$ has length 2. Hence, we can see that x can be built up from the empty string by applying the recursive step with x_1, then x_2, and so on up to x_n, which shows that $x \in S$.

(b) Binary strings not containing 10.

Solution:

If the string does not contain 10, then the first 1 in the string can only be followed by more 1s. Hence, it must be of the form 0^m1^n for some $m, n \in \mathbb{N}$.

Basis: $\varepsilon \in S$.

Recursive Step: If $x \in S$, then $0x \in S$ and $x1 \in S$.

Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

Brief Justification: The empty string satisfies the property, and the recursive step cannot place a 0 after a 1 since it only adds 0s on the left. Hence, every string in S satisfies the property.

In the other direction, from our discussion above, any string of this form can be written as $y = 0^m1^n$ for some $m, n \in \mathbb{N}$. We can build up the string y from the empty string by applying the rule $x \to 0x$ m times and then applying the rule $x \to x1$ n times. This shows that the string y is in S.

(c) Binary strings not containing 10 as a substring and having at least as many 1s as 0s.

Solution:

These must be of the form 0^m1^n for some $m, n \in \mathbb{N}$ with $m \leq n$. We can ensure that by pairing up the 0s with 1s as they are added:

Basis: $\varepsilon \in S$.

Recursive Step: If $x \in S$, then $0x1 \in S$ and $x1 \in S$.

Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

Brief Justification: As in the previous part, we cannot add a 0 after a 1 because we only add 0s at the front. And since every 0 comes with a 1, we always have at least as many 1s as 0s.

In the other direction, from our discussion above, any string of this form can be written as xy, where $x = 0^m1^m$ and $y = 1^{n-m}$, since $n \geq m$. We can build up the string x from the empty string by applying the rule $x \to 0x1$ m times and then produce the string xy by applying the rule $x \to x1$ $n-m$ times, which shows that the string is in S.

(d) Binary strings containing at most two 0s and at most two 1s.

Solution:

This is the set of all binary strings of length at most 4 *except* for these:

$$000, 1000, 0100, 0010, 0001, 0000, 111, 0111, 1011, 1101, 1110, 1111$$

Since this is a **finite set**, we can define it recursively using only basis elements and no recursive step.