Section 03: Propositions and Proofs

1. Domain Restriction

Translate each of the following sentences into logical notation. These translations require some of our quantifier tricks. You may use the operators $+$ and $\cdot$ which take two numbers as input and evaluate to their sum or product, respectively. Remember:

- To restrict the domain under a $\forall$ quantifier, add a hypothesis to an implication.
- To restrict the domain under an $\exists$ quantifier, AND in the restriction.
- If you want variables to be different, you have to explicitly require them to be not equal.

(a) Domain: Positive integers; Predicates: `Even`, `Prime`, `Equal`
"There is only one positive integer that is prime and even."

(b) Domain: Real numbers; Predicates: `Even`, `Prime`, `Equal`
"There are two different prime numbers that sum to an even number."

(c) Domain: Real numbers; Predicates: `Even`, `Prime`, `Equal`
"The product of two distinct prime numbers is not prime."

(d) Domain: Real numbers; Predicates: `Even`, `Prime`, `Equal`, `Postivite`, `Greater`, `Integer`
"For every positive integer, there is a greater even integer"

2. Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

(a) $\forall x\ \forall y\ P(x,y)$ $\qquad$ $\forall y\ \forall x\ P(x,y)$

(b) $\exists x\ \exists y\ P(x,y)$ $\qquad$ $\exists y\ \exists x\ P(x,y)$

(c) $\forall x\ \exists y\ P(x,y)$ $\qquad$ $\forall y\ \exists x\ P(x,y)$

(d) $\forall x\ \exists y\ P(x,y)$ $\qquad$ $\exists x\ \forall y\ P(x,y)$

(e) $\forall x\ \exists y\ P(x,y)$ $\qquad$ $\exists y\ \forall x\ P(x,y)$

3. Quantifier Ordering

Let your domain of discourse be a set of `Element` objects given in a list called `Domain`. Imagine you have a predicate `pred`(x,y), which is encoded in the java method `public boolean pred(int x, int y)`. That is you call your predicate `pred` true if and only if the java method returns `true`.

(a) Consider the following Java method:

```
public boolean Mystery(Domain D){
    for(Element x : D) {
        for(Element y : D) {
```

```
            if(pred(x,y))
                return true;
        }
    }
}
```

Mystery corresponds to a quantified formula (for D being the domain of discourse), what is that formula?

(b) What formula does mystery2 correspond to

```
public boolean Mystery2(Domain D){
    for(Element x : D) {
        boolean thisXPass = false;
        for(Element y : D) {
            if(pred(x,y))
                thisXPass = true;
        }
        if(!thisXPass)
            return false;
    }
    return true;
}
```

4. Find the Bug

Each of these inference proofs is incorrect. Identify the line (or lines) which incorrectly apply a law, and explain the error. Then, if the claim is false, give concrete examples of propositions to show it is false. If it is true, write a correct proof.

(a) This proof claims to show that given $a \to (b \vee c)$, we can conclude $a \to c$.

1.	$a \to (b \vee c)$	[Given]
2.1.	a	[Assumption]
2.2.	$\neg b$	[Assumption]
2.3.	$b \vee c$	[Modus Ponens, from 1 and 2.1]
2.4.	c	[$\vee$ elimination, from 2.2 and 2.3]
2.	$a \to c$	[Direct Proof Rule, from 2.1-2.4]

(b) This proof claims to show that given $p \to q$ and r, we can conclude $p \to (q \vee r)$.

$1.p \to q$	[Given]
$2.r$	[Given]
$3.p \to (q \vee r)$	[Intro $\vee$ (1,2)]

(c) This proof claims to show that given $p \to q$ and q that we can conclude p

$1.p \to q$	[Given]
$2.q$	[Given]
$3.\neg p \vee q$	[Law of Implication (1)]
$4.q$	[Eliminate $\vee$ (2,3)

5. Formal Spoofs

For each of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $\exists z\ \forall x\ P(x,z)$ follows from $\forall x\ \exists y\ P(x,y)$.

1.	$\forall x\ \exists y\ P(x,y)$	[Given]
2.	$\forall x\ P(x,c)$	[$\exists$ Elim: 1, c special]
3.	$\exists z\ \forall x P(x,z)$	[$\exists$ Intro: 2]

(b) Show that $\exists z\ (P(z) \wedge Q(z))$ follows from $\forall x\ P(x)$ and $\exists y\ Q(y)$.

1.	$\forall x\ P(x)$	[Given]
2.	$\exists y\ Q(y)$	[Given]
3.	Let z be arbitrary	
4.	$P(z)$	[$\forall$ Elim: 1]
5.	$Q(z)$	[$\exists$ Elim: 2, let z be the object that satisfies $Q(z)$]
6.	$P(z) \wedge Q(z)$	[$\wedge$ Intro: 4, 5]
7.	$\exists z\ P(z) \wedge Q(z)$	[$\exists$ Intro: 6]

6. Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \vee q$, $q \rightarrow r$ and $r \rightarrow s$.

7. Predicate Logic Formal Proof

Given $\forall x. T(x) \rightarrow M(x)$, we wish to prove $(\exists x. T(x)) \rightarrow (\exists y. M(y))$. The following formal proof does this, but it is missing citations for which rules are used, and which lines they are based on. Fill in the blanks with inference rules or predicate logic equivalences, as well as the line numbers.

Then, summarize in English what is going on here.

1.	$\forall x. T(x) \rightarrow M(x)$	(________________)
2.1.	$\exists x. T(x)$	(________________)
	Let r be the object that satisfies $T(r)$	
2.2.	$T(r)$	(________________, from ________)
2.3.	$T(r) \rightarrow M(r)$	(________________, from ________)
2.4.	$M(r)$	(________________, from ________)
2.5.	$\exists y. M(y)$	(________________, from ________)
2.	$(\exists x. T(x)) \rightarrow (\exists y. M(y))$	(________________, from ________)

8. Formal Proof

Show that $\neg p$ follows from $\neg(\neg r \vee t)$, $\neg q \vee \neg s$ and $(p \rightarrow q) \wedge (r \rightarrow s)$.

9. A Formal Proof in Predicate Logic

Prove $\exists x\ (P(x) \vee R(x))$ from $\forall x\ (P(x) \vee Q(x))$ and $\forall y\ (\neg Q(y) \vee R(y))$.