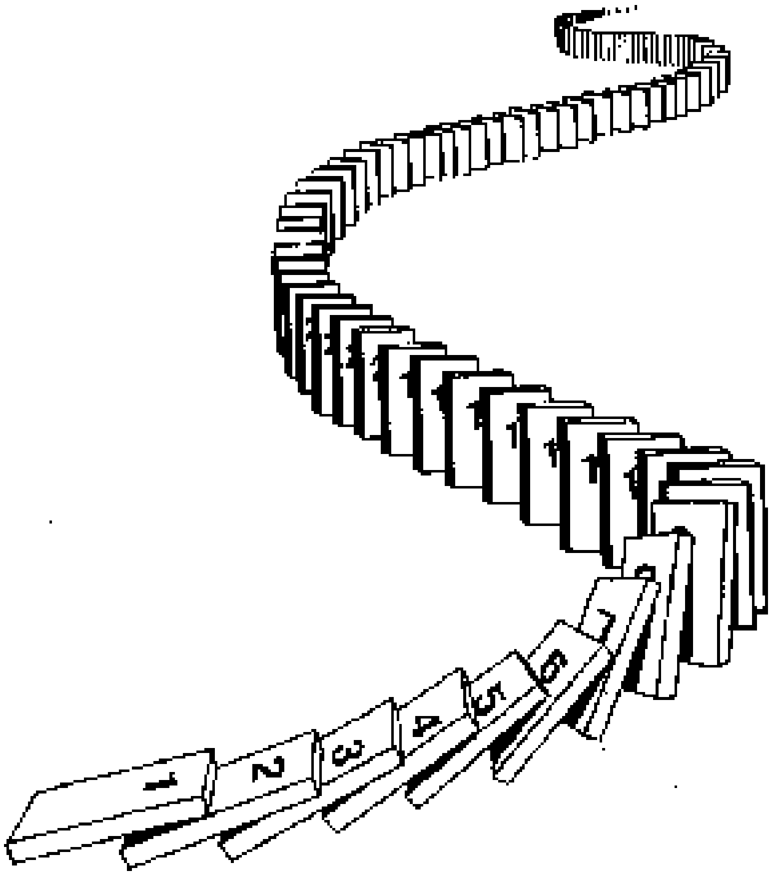
CSE 311: Foundations of Computing

Lecture 17: Strong induction



Recap from last lecture

The induction inference rule:

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Recap from last lecture

The induction inference rule:

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

The induction template in an English proof:

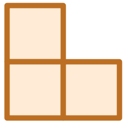
Proof:

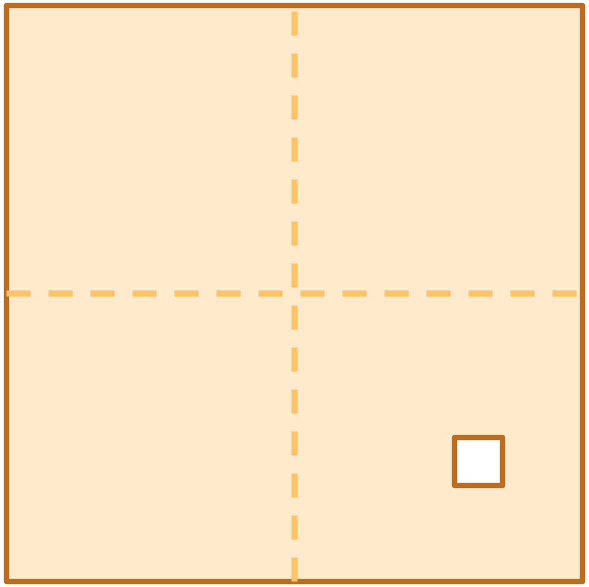
1. "Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

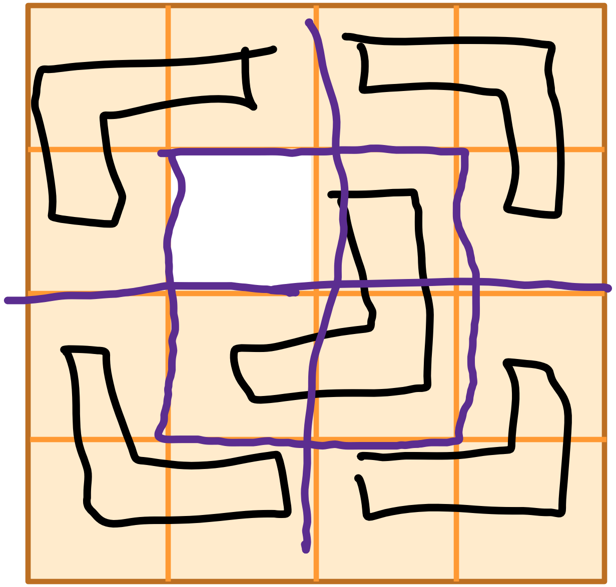
   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

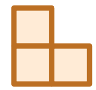
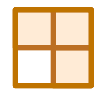
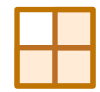
Checkerboard Tiling

- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 





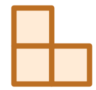
Checkerboard Tiling

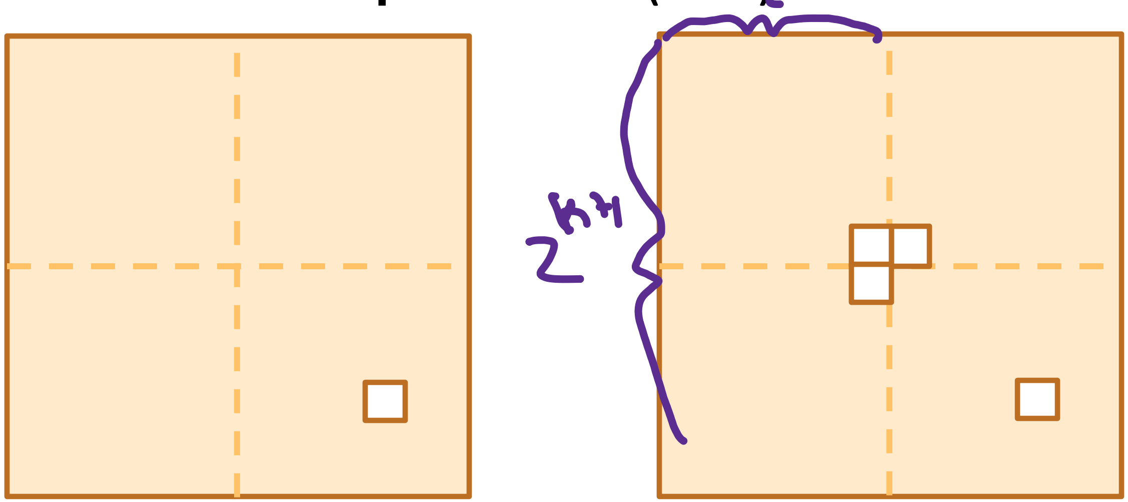
1. Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with  .
   We prove $P(n)$ for all $n \geq 1$ by induction on n.
2. Base Case: $n=1$    

Checkerboard Tiling

Define $\underbrace{Q(n)}=P(n+1)$

$Q(0)=P(1)$

1. Let $P(n)$ be any $2^n \times 2^n$ checkerboard with one square removed can be tiled with  .
   We prove $P(n)$ for all $n \geq 1$ by induction on n.
2. **Base Case:** $n=1$ 
3. **Inductive Hypothesis:** Assume $P(k)$ for some arbitrary integer $k \geq 1$
4. **Inductive Step:** Prove $P(k+1)$ 2^k



Apply IH to each quadrant then fill with extra tile.

Prove $3^n \geq n^2 + 3$ **for all** $n \geq 2$

$3^2 = 9$

$2^2 + 3 = 7$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

$3^2 \geq 2^2+3$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$. **I.e., suppose** $3^k \geq k^2+3$**.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**. I.e., suppose** $3^k \geq k^2+3$**.**
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**. I.e., suppose** $3^k \geq k^2+3$**.**
4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

$3^2 = 9 \geq 7 = 4+3 = 2^2+3$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers $n \geq 2$ by induction.
2. Base Case (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so P(2) is true.
3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 2$. I.e., suppose $3^k \geq k^2+3$.
4. Inductive Step:

Goal: Show P(k+1), i.e. show $3^{k+1} \geq (k+1)^2+3 = k^2+2k+4$

$x^a x^b = x^{a+b}$

$3^1 \cdot 3^k = 3^{1+k}$

$3^{k+1} = 3(3^k)$

$\geq 3(k^2+3)$ by the IH

$= 3k^2+9$

$= k^2+2k^2+9$

$\geq k^2+2k+4 = (k+1)^2+3$ since $k \geq 1$.

$A = B$

$\geq C$

$= D$

$A = B$

$B \geq C$

$C = D$

$\therefore A \geq D$

Therefore P(k+1) is true.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**. I.e., suppose** $3^k \geq k^2+3$**.**
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3^{k+1} \geq (k+1)^2+3=k^2+2k+4}$

   $$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(k^2+3) \text{ by the IH} \\ &= k^2+2k^2+9 \\ &\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 1. \end{aligned}$$

   Therefore P(k+1) **is true.**
5. **Thus** P(n) **is true for all integers** $n \geq 2$**, by induction.**

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$\begin{tikzcd}
P(0) \arrow[r, bend left, "P(0)\to P(1)"] & P(1) \arrow[r, bend left, "P(1)\to P(2)"] & P(2) \arrow[r, bend left, "P(2)\to P(3)"] & P(3) \arrow[r, bend left, "P(3)\to P(4)"] & P(4) \arrow[r, bend left, "P(4)\to P(5)"] & P(5)
\end{tikzcd}$$

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$\begin{tikzcd} P(0) \arrow[r, bend left, "P(0)\to P(1)"] & P(1) \arrow[r, bend left, "P(1)\to P(2)"] & P(2) \arrow[r, bend left, "P(2)\to P(3)"] & P(3) \arrow[r, bend left, "P(3)\to P(4)"] & P(4) \arrow[r, bend left, "P(4)\to P(5)"] & P(5) \end{tikzcd}$$

We made it harder than we needed to ...

- **When we proved** $P(2)$ **we knew BOTH** $P(0)$ **and** $P(1)$
- **When we proved** $P(3)$ **we knew** $P(0)$ **and** $P(1)$ **and** $P(2)$
- **When we proved** $P(4)$ **we knew** $P(0)$, $P(1)$, $P(2)$, $P(3)$
- **etc.**

That's the essence of the idea of Strong Induction.

Strong Induction

$$P(0)$$

$$\forall k \ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \rightarrow P(k+1)\Big)$$

$$\therefore \forall n \, P(n)$$

Strong Induction

$$P(0)$$
$$\forall k\ \Big(\big(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)\big) \rightarrow P(k+1)\Big)$$

$$\therefore \forall n\ P(n)$$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)$$

Note that $Q(0) \equiv P(0)$ and $Q(k+1) \equiv Q(k)\ \wedge P(k+1)$ and $\forall n\ Q(n) \equiv \forall n\ P(n)$

Inductive Proofs In 5 Easy Steps

1. **"Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."**
2. **"Base Case:" Prove $P(b)$**
3. **"Inductive Hypothesis:**

   Assume that for some arbitrary integer $k \geq b$,

   $P(k)$ is true"
4. **"Inductive Step:" Prove that $P(k+1)$ is true:**

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. **"Conclusion: $P(n)$ is true for all integers $n \geq b$"**

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$, and maybe $P(b+1)$, $P(b+2)$ etc.
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Lecture 17 Activity

You will be assigned to breakout rooms. Please:

- Introduce yourself
- Choose someone to share their screen, showing this PDF
- Prove that any amount $\geq \$12$ can be paid with coins of value \$4 & \$5. Fill in the gaps in the following proof:

1. "Set $P(n) :=$ ``there are integers s, t with $s \geq 0, t \geq 0$ so that $n = 4s + 5t$''. We will show that $P(n)$ is true for all integers $n \geq 12$ by *strong* induction."
2. "Base Case:" Prove $P(12), P(13), P(14), P(15)$. ......
3. "Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 15$, *$P(j)$ is true for every integer j from 12 to k*"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   ....
5. "Conclusion: $P(n)$ is true for all integers $n \geq 12$"

Fill out the poll everywhere for Activity Credit!
Go to pollev.com/philipmg and login with your UW identity

Lecture 17 Activity

You will be assigned to breakout rooms. Please:

- Prove that any amount $\geq \$12$ can be paid with coins of value \$4 & \$5. Fill in the gaps in the following proof:

1. "Set $P(n) :=$ ``there are integers s, t with $s \geq 0, t \geq 0$ so that $n = 4s + 5t$''. We will show that $P(n)$ is true for all integers $n \geq 12$ by *strong* induction."
2. "Base Case:" Prove $P(12), P(13), P(14), P(15)$.

   $P(12)$ holds because $12 = 4 \cdot 3 + 5 \cdot 0$

   $P(13)$ holds because $13 = 4 \cdot 2 + 5 \cdot 1$

   $P(14)$ holds because $14 = 4 \cdot 1 + 5 \cdot 2$

   $P(15)$ holds because $15 = 4 \cdot 0 + 5 \cdot 3$
3. "Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 15$, *$P(j)$ is true for every integer j from* 12 to k"

   $P(12) \wedge P(13) \wedge \ldots \wedge P(k)$
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   We know that $k + 1 - 4 \geq 12$ and $k + 1 - 4 \leq k$. Hence $P(k + 1 - 4)$ is true by IH. Then there are non-negative integers s, t with $k + 1 - 4 = 4s + 5t$. That means $k + 1 = 4 \cdot (s + 1) + 5t$. Hence $P(k+1)$ is true.
5. "Conclusion: $P(n)$ is true for all integers $n \geq 12$"

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$**,**
   P(j) **is true for every integer** j **between** 2 **and** k

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$**,**
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$**,**
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; i.e. } k+1 \textbf{ is a product of primes}}$

   <u>Case: k+1 **is prime**</u>: **Then by definition** k+1 **is a product of primes**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** k ≥ 2,
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; i.e. } k+1 \textbf{ is a product of primes}}$

   <u>**Case:** k+1 **is prime**</u>**: Then by definition** k+1 **is a product of primes**

   <u>**Case:** k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b **where** 2 ≤ a, b ≤ k.

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$,
   $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show $P(k+1)$; **i.e.** $k+1$ **is a product of primes**

   <u>Case: $k+1$ is prime</u>: Then by definition $k+1$ **is a product of primes**

   <u>Case: $k+1$ is composite:</u> Then $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** $P(a)$ **and** $P(b)$ **are true so we have**

   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

   Since $k \geq 1$, **one of these cases must happen and so** $P(k+1)$ **is true.**

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$**,**
   $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   $\boxed{\text{Goal: Show } P(k+1)\text{; i.e. } k+1 \text{ is a product of primes}}$

   <u>Case: $k+1$ is prime</u>: **Then by definition** $k+1$ **is a product of primes**

   <u>Case: $k+1$ is composite:</u> **Then** $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH**, $P(a)$ **and** $P(b)$ **are true so we have**

   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r$, $q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

   Since $k \geq 2$**, one of these cases must happen and so** $P(k+1)$ **is true.**
5. **Thus** $P(n)$ **is true for all integers** $n \geq 2$**, by strong induction.**

Strong Induction is particularly useful when…

…we need to analyze methods that on input k make a recursive call for an input different from $k-1$.

e.g.: Recursive Modular Exponentiation:

- For exponent $k > 0$ it made a recursive call with exponent $\mathrm{j} = k/2$ when k was even or $\mathrm{j} = k-1$ when k was odd.

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0$; $F(n+1) = F(n) + 1$ **for all** $n \geq 0$.

- $G(0) = 1$; $G(n+1) = 2 \cdot G(n)$ **for all** $n \geq 0$.

- $0! = 1$; $(n+1)! = (n+1) \cdot n!$ **for all** $n \geq 0$.

- $H(0) = 1$; $H(n+1) = 2^{H(n)}$ **for all** $n \geq 0$.

Prove $n! \leq n^n$ for all $n \geq 1$

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "n! ≤ n^n". **We will show that** P(n) **is true for all integers** n ≥ 1 **by induction.**
2. **Base Case** (n=1): $1!=1\cdot 0!=1\cdot 1=1=1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 1. **I.e., suppose** $k! \leq k^k$.
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } (k+1)! \leq (k+1)^{k+1}}$

   $$\begin{aligned} (k+1)! &= (k+1)\cdot k! && \textbf{by definition of !} \\ &\leq (k+1)\cdot k^k && \textbf{by the IH and } k+1>0 \\ &\leq (k+1)\cdot (k+1)^k && \textbf{since } k \geq 0 \\ &= (k+1)^{k+1} \end{aligned}$$

   Therefore P(k+1) **is true.**
5. **Thus** P(n) **is true for all** n ≥ 1, **by induction.**