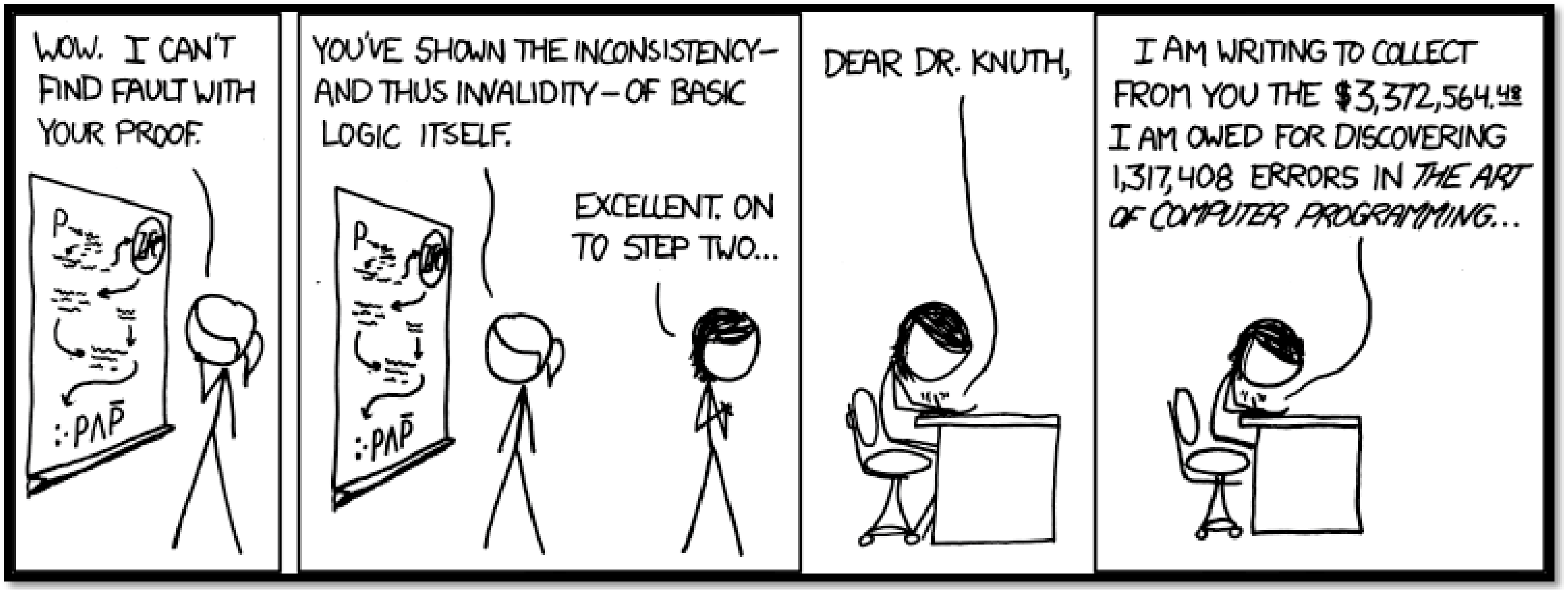
CSE 311: Foundations of Computing

Lecture 7: Logical Inference



Recap from last lecture: Logical inference

- <u>Given:</u> A list of (predicate/prop. logic) formulas as **facts**.
- <u>Question:</u> What other facts can be derived from those?

- Our first inference rule: **Modus Ponens**
  Application: If our list of given facts includes both *q* and *r* then we can infer that also *r* is true.

- Modus Ponens is written in compact form as

$$\begin{array}{c} \underline{q,\ q \rightarrow r} \\ \therefore\ r \end{array}$$

Axioms: Special inference rules

If I have nothing...

Requirements: ______________

Conclusions: $\therefore$ C , D

Then, C must be true

Then D must be true

Example (Excluded Middle):

$$\frac{}{\therefore\ A \lor \neg A}$$

$A \lor \neg A$ must be true.

My First Proof!

Show that s follows from q, $q \rightarrow r$, and $r \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.	$r \rightarrow s$	Given
4.		
5.		

My First Proof!

Show that s follows from q, $q \rightarrow r$, and $r \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.	$r \rightarrow s$	Given
4.	r	MP: 1, 2
5.	s	MP: 3, 4

Proofs can use equivalences too

Show that $\neg q$ follows from $q \rightarrow r$ and $\neg r$

1.	$q \rightarrow r$	Given
2.	$\neg r$	Given
3.		
4.		

Proofs can use equivalences too

Show that $\neg q$ follows from $q \rightarrow r$ and $\neg r$

1.	$q \rightarrow r$	Given
2.	$\neg r$	Given
3.	$\neg r \rightarrow \neg q$	Contrapositive: 1
4.		

Proofs can use equivalences too

Show that $\neg q$ follows from $q \rightarrow r$ and $\neg r$

1.	$q \rightarrow r$	Given
2.	$\neg r$	Given
3.	$\neg r \rightarrow \neg q$	Contrapositive: 1
4.	$\neg q$	MP: 2, 3

Simple Propositional Inference Rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

∧ introduction

∧ Elimination

$$\frac{q \wedge r}{\therefore q, r}$$

$$\frac{q, r}{\therefore q \wedge r}$$

[link] Reference sheet with all inference rules

∨ Elimination

$$\frac{q \vee r, \neg q}{\therefore r}$$

∨ intro

$$\frac{q}{\therefore q \vee r}$$

MP

$$\frac{q, q \rightarrow r}{\therefore r}$$

$$\frac{q \Rightarrow r}{\therefore q \rightarrow r}$$

Direct Proof Rule

Not like other rules

\Rightarrow implies "proves"

Proofs

Show that s follows from q, $q \to r$ and $(q \land r) \to s$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{q,\ q \to r}{\therefore\ r}$$

$$\frac{q \land r}{\therefore\ q,\ r}$$

$$\frac{q,\ r}{\therefore\ q \land r}$$

Proofs

Show that s follows from $q, q \rightarrow r,$ and $(q \land r) \rightarrow s$

1.
2.
3.
4.
5.
6.

Proofs

Show that s follows from $q, q \rightarrow r$, and $(q \wedge r) \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.		
4.		
5.		
6.		

Proofs

Show that s follows from $q, q \rightarrow r$, and $(q \wedge r) \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.	r	MP: 1, 2
4.		
5.		
6.		

Proofs

Show that s follows from $q, q \rightarrow r,$ and $(q \land r) \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.	r	MP: 1, 2
4.	$q \land r$	Intro $\land$: 1, 3
5.		
6.		

Proofs

Show that s follows from $q, q \to r,$ and $(q \land r) \to s$

1.	q	Given
2.	$q \to r$	Given
3.	r	MP: 1, 2
4.	$q \land r$	Intro $\land$: 1, 3
5.	$(q \land r) \to s$	Given
6.		

Proofs

Show that s follows from $q, q \to r$, and $(q \land r) \to s$

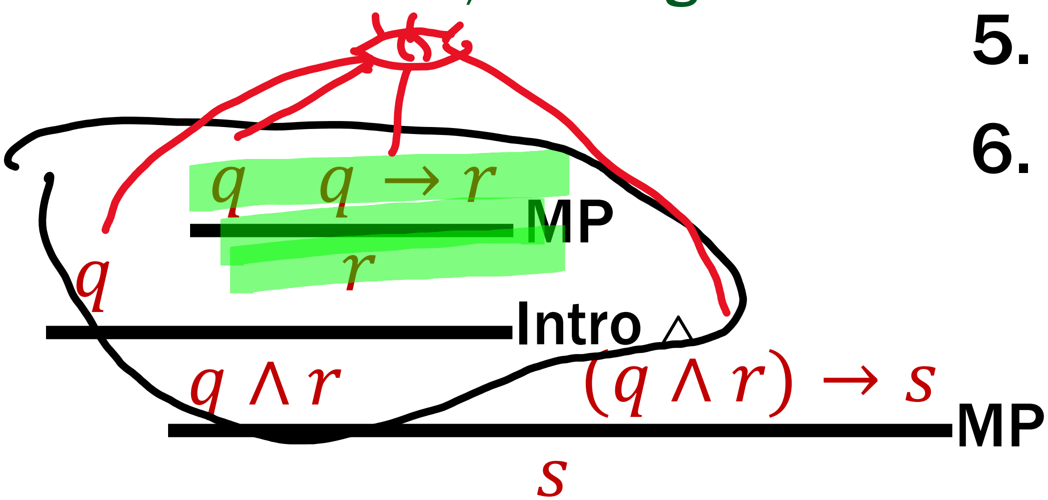
1.	q	Given
2.	$q \to r$	Given
3.	r	MP: 1, 2
4.	$q \land r$	Intro $\land$: 1, 3
5.	$(q \land r) \to s$	Given
6.	s	MP: 4, 5

Proofs

Show that s follows from $q, q \rightarrow r$, and $(q \wedge r) \rightarrow s$

1.	q	Given
2.	$q \rightarrow r$	Given
3.	r	MP: 1, 2
4.	$q \wedge r$	Intro $\wedge$: 1, 3
5.	$(q \wedge r) \rightarrow s$	Given
6.	s	MP: 4, 5

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!



Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

  $A \wedge (B \rightarrow C) \equiv A \wedge (\neg B \vee C)$

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $q \rightarrow r$ given

~~2. $(q \vee s) \rightarrow r$ intro $\vee$ from 1.~~

$F \rightarrow F \equiv T$

$(F \vee T) \rightarrow F \equiv F$

Does not follow! e.g . q=**F**, r=**F**, s=**T**

Lecture 7 Activity

- You will be assigned to **breakout rooms**. Please:
- Introduce yourself
- Choose someone to share screen, showing this PDF
- Suppose you are given $p \rightarrow q$, $\neg s \rightarrow \neg q$ and p as facts. Find a sequence of inference rules that show that then s is true.

Fill out a poll everywhere for Activity Credit!
Go to pollev.com/philipmg and login with your UW identity

Lecture 7 Activity

Given $p \to q$, $\neg s \to \neg q$, p

Prove: s

$A \to B \equiv \neg B \to \neg A$

1. $p \to q$ Given
2. $\neg s \to \neg q$ Given
3. p Given
4. q MP 1,3
5. $q \to s$ Contrapositive 2
6. s MP 4.5

Proofs

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

First: Write down givens and goal

20. $\neg r$?

Idea: Work backwards!

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
20.	$\neg r$	MP: 2, ?

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim →" which is MP.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove q...
  - Notice that at this point, if we prove q, we've proven $\neg r$...

Proofs

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
19.	q	?
20.	$\neg r$	MP: 2, 19

This looks like or-elimination.

Elim ∨ $\dfrac{A \vee B \,;\, \neg A}{\therefore B}$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
18.	$\neg\neg s$	?
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

$\neg\neg s$ doesn't show up in the givens but s does and we can use equivalences

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	?
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \rightarrow \neg r$	Given
3.	$\neg s \vee q$	Given
17.	s	$\wedge$ Elim: 1
18.	$\neg\neg s$	Double Negation: 17
19.	q	$\vee$ Elim: 3, 18
20.	$\neg r$	MP: 2, 19

No holes left! We just need to clean up a bit.

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.

1.	$p \wedge s$	Given
2.	$q \to \neg r$	Given
3.	$\neg s \vee q$	Given
4.	s	$\wedge$ Elim: 1
5.	$\neg\neg s$	Double Negation: 4
6.	q	$\vee$ Elim: 3, 5
7.	$\neg r$	MP: 2, 6

To Prove An Implication: $A \rightarrow B$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is **a proof** that "**Given** A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \rightarrow B$ is true

Example: Prove $q \rightarrow (q \vee r)$.

proof subroutine

Indent proof subroutine $\Rightarrow$

1.	q	Assumption
2.	$q \vee r$	Intro $\vee$: 1
3.	$q \rightarrow (q \vee r)$	Direct Proof Rule

Proofs using the direct proof rule

Show that $q \rightarrow s$ follows from r and $(q \wedge r) \rightarrow s$

1. r Given
2. $(q \wedge r) \rightarrow s$ Given

This is a proof of $q \rightarrow s$

- 3.1. q Assumption
- 3.2. $q \wedge r$ Intro $\wedge$: 1, 3.1
- 3.3. s MP: 2, 3.2

If we know q is true… Then, we've shown s is true

3. $q \rightarrow s$ Direct Proof Rule

Example

Prove: $(q \wedge r) \rightarrow (q \vee r)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens…

Example

Prove: $(q \wedge r) \rightarrow (q \vee r)$

Example

Prove: $(q \wedge r) \rightarrow (q \vee r)$

1.1.	$q \wedge r$	Assumption
1.2.	q	Elim $\wedge$: 1.1
1.3.	$q \vee r$	Intro $\vee$: 1.2
1.	$(q \wedge r) \rightarrow (q \vee r)$	Direct Proof Rule

Example

Prove: $((q \rightarrow r) \wedge (r \rightarrow s)) \rightarrow (q \rightarrow s)$

Example

Prove: $((q \rightarrow r) \wedge (r \rightarrow s)) \rightarrow (q \rightarrow s)$

1.1.	$(q \rightarrow r) \wedge (r \rightarrow s)$	Assumption
1.2.	$q \rightarrow r$	∧ Elim: 1.1
1.3.	$r \rightarrow s$	∧ Elim: 1.1
1.4.1.	q	Assumption
1.4.2.	r	MP: 1.2, 1.4.1
1.4.3.	s	MP: 1.3, 1.4.2
1.4.	$q \rightarrow s$	Direct Proof Rule
1.	$((q \rightarrow r) \wedge (r \rightarrow s)) \rightarrow (q \rightarrow s)$	Direct Proof Rule

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.