Homework 3: Predicate Logic

Changelog: This is version 2, updated April 14 at 9 PM. Simplified wording of 2 and 3.

Due date: Wednesday April 21 at 11:00 PM (Seattle time, i.e. GMT-7)
If you work with others (and you should!), remember to follow the collaboration policy.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting.

Be sure to read the grading guidelines for more information on what we're looking for.

This homework has 5 pages, make sure you keep scrolling!

While working on the assignment, please (roughly) keep track of the time you spent on each problem. This is so that you have an idea of how to answer the feedback questions!

1. Collaborators

List any collaborators (i.e., other students) you worked with and which problems you worked on together, or state that you worked alone.

2. Nested Quantifiers [10 points]

Fix the domain of discourse to be all "groups," with the following predicates.[1]

- $\texttt{subgroup}(x, y)$ is true when x is a subgroup of y.
- $\texttt{isomorphic}(x, y)$ is true when the groups x and y are isomorphic to each other.
- $\texttt{symmetricGroup}(x, y)$ is true when y is the symmetric group of x.

(a) Your friend tried to translate "There is a group that is isomorphic to a subgroup of every symmetric group" and got: "$\exists a \forall b \forall c.\ \texttt{symmetricGroup}(b, c) \rightarrow \texttt{subgroup}(a, c)$". The translation is incorrect. Give a correct translation, and describe a scenario (i.e. facts about groups and functions) in which your translation and their translation evaluate to different truth values.

(b) Translate the following sentence into predicate logic: "Every group G is isomorphic to a subgroup of the symmetric group of G."[2]

[1]A group is a formally-defined mathematical object, but you don't need to know anything about groups for this course.
[2]This is the statement of Cayley's Theorem, from group theory.

3. Resolution [12 points]

(a) Given all of the following statements show that you can infer `F` [10 points].

- $A \lor B$
- $\neg B \lor C$
- $C \lor \neg A$
- $A \lor \neg B \lor \neg C$
- $\neg A \lor B \lor \neg C$
- $\neg A \lor \neg B \lor \neg C$

Hint. The *resolution inference rule* is of the form $\dfrac{q \lor r, s \lor \neg r}{\therefore q \lor s}$ and you may use it in this exercise.

(b) What is your conclusion about these propositional logic formulas? [2 points]

4. For every iteration [6 points]

Imagine you have the predicate $\texttt{pred}(x, y)$, which is true if and only if the java method `public boolean pred(Element x, Element y)` returns true. Write a java method that takes in a `Domain` object (which is a list of all the `Elements` in the domain) and returns the value of $\exists x \forall y\, \texttt{pred}(x, y)$

You do not need to follow 142/143's style rules for code, but if your code is extremely unnecessarily convoluted you may lose points. We won't grade your code for java details (e.g. if you forget a semicolon, but it's clear what you meant we won't deduct; but errors that affect our understanding [say forgetting braces] may lead to deductions). You may want to consult Section 3's handout for examples of this type of code. If you're working in LaTeX you may want to use the verbatim environment (or just code in a text editor and insert a picture). If you are not familiar with java, clearly written pseudocode is okay as well.

5. Spoof [14 points]

Theorem: Given $a \land \neg b$, $r \to s$, and $a \to \neg(\neg b \land s)$ prove $\neg r$.
"Spoof":

1.$a \land \neg b$	Given
2.a	$\land$ Elim: 1
3.$a \to \neg(\neg b \land s)$	Given
4.$\neg(\neg b \land s)$	MP: 2,3
5.$\neg\neg b \land \neg s$	DeMorgan's: 4
6.$b \land \neg s$	Double negation: 5
7.$\neg s$	$\land$ Elim: 6
8.$r \to s$	Given
9.$\neg s \to \neg r$	Contrapositive: 8
10.$\neg r$	MP: 7,9

(a) What is the most significant error in this proof? Give the line and briefly explain why it is wrong. [5 points]

(b) Show the theorem is true by fixing the error in the spoof. For this problem, please entirely rewrite the proof in your submission. [9 points]

6. Find The Bug [16 points]

The following proof claims to show that
Given: $\exists x P(x) \wedge \exists x Q(x)$, $\forall x(Q(x) \rightarrow R(x))$
Prove: $\exists x(P(x) \wedge R(x))$

1.$\exists x P(x) \wedge \exists x Q(x)$	Given
2.$P(c) \wedge Q(c)$	Eliminate $\exists$ (1)
3.$P(c)$	Eliminate $\wedge$ (2)
4.$Q(c)$	Eliminate $\wedge$ (2)
5.$\forall x(Q(x) \rightarrow R(x))$	Given
6.$Q(c) \rightarrow R(c)$	Eliminate $\forall$
7.$R(c)$	Modus Ponens (6,4)
8.$\exists x P(x)$	Introduce $\exists$
9.$\exists x R(x)$	Introdcue $\exists$
10.$\exists x(P(x) \wedge R(x))$	Introduce $\wedge$

(a) There is a bug in steps 1-4, where a rule is applied in a way that is not allowed. Identify the line where the rule is applied incorrectly, and explain why it is incorrect. [5 points]

(b) There is a bug in steps 5-10, **ignoring any mistakes that had happened before**, where a rule is applied incorrectly. Identify the line where this rule is applied incorrectly, and explain why it is incorrect. [5 points]

(c) Is the claim true?
If it is true, describe how to correct the proof. (You may say things like "replace step 3 with..." or "insert the following between steps 6 and 7..." or, if you prefer, you may rewrite the whole proof.
If the claim is false, describe P, Q, R and a domain of discourse such that the givens are true but the thing to prove is false. [6 points]

7. Inference Proof [20 points]

Theorem: Given $s \to (t \wedge q)$, $\neg s \to r$, and $(r \vee t) \to q$, prove q.

"Spoof:"

1.	$\neg s \to r$	[Given]
2.	$(r \vee t) \to q$	[Given]
3.	$r \to q$	[Elim of $\vee$: 2]
	4.1. $\neg s$ [Assumption]	
	4.2. r [MP: 4.1, 1]	
	4.3. q [MP: 4.2, 3]	
4.	$\neg s \to q$	[Direct Proof Rule]
	5.1. s [Assumption]	
	5.2. $s \to (t \wedge q)$ [Given]	
	5.3. $t \wedge q$ [MP: 5.1, 5.2]	
	5.4. q [Elim of $\wedge$: 5.3]	
5.	$s \to q$	[Direct Proof Rule]
6.	$(s \to q) \wedge (\neg s \to q)$	[Intro $\wedge$: 5, 4]
7.	$(\neg s \vee q) \wedge (\neg\neg s \vee q)$	[Law of Implication]
8.	$(\neg s \vee q) \wedge (s \vee q)$	[Double Negation]
9.	$((\neg s \vee q) \wedge s) \vee ((\neg s \vee q) \wedge q)$	[Distributivity]
10.	$((\neg s \vee q) \wedge s) \vee (q \wedge (\neg s \vee q))$	[Commutativity]
11.	$((\neg s \vee q) \wedge s) \vee (q \wedge (q \vee \neg s))$	[Commutativity]
12.	$((\neg s \vee q) \wedge s) \vee q$	[Absorption]
13.	$(s \wedge (\neg s \vee q)) \vee q$	[Commutativity]
14.	$((s \wedge \neg s) \vee q) \vee q$	[Associativity]
15.	$(F \vee q) \vee q$	[Negation]
16.	$(q \vee F) \vee q$	[Commutativity]
17.	$q \vee q$	[Identity]
18.	q	[Idempotence]

(a) There are two major errors in this proof. Indicate which lines contain the errors and, for each one, explain (as briefly as possible) why that line is incorrect. [8 points]

(b) Is the conclusion of the "spoof" correct? If it is incorrect, describe propositions t, q, r, s such that the givens are true, but the claim is false. If the conclusion is correct, briefly explain how to correct any errors in lines 1–5 (you'll explain errors in 6–18 in part c). [4 points]

(c) Give a correct proof of what is claimed in lines 6–18, i.e., that from $(s \to q) \wedge (\neg s \to q)$, we can infer that q is true. [8 points]

8. Inference Proof with Quantifiers [12 points]

Using the logical inference rules and equivalences we have given, write an *inference proof* that given $\forall x([\exists y P(x, y)] \to \neg Q(x))$, $\forall x(\neg R(x) \to (Q(x) \vee \neg P(x, x)))$, and $\exists x\ P(x, x)$, you can conclude that $\exists x\ R(x)$.

You should consult the symbolic proof guidelines for our expectations on these proofs.

9. Extra Credit: Space Pirates

Five pirates, called Ann, Brenda, Carla, Danielle and Emily, found a treasure of 100 gold coins. On their ship, they decide to split the coins using the following scheme:

- The first pirate in alphabetical order becomes the chief pirate.
- The chief proposes how to share the coins, and all other pirates (excluding the chief) vote for or against it.
- If 50% or more of the pirates vote for it, then the coins will be shared that way.
- Otherwise, the chief will be thrown overboard, and the process is repeated with the pirates that remain.

Thus, in the first round Ann is the chief: if her proposal is rejected, she is thrown overboard and Brenda becomes the chief, etc; if Ann, Brenda, Carla, and Danielle are thrown overboard, then Emily becomes the chief and keeps the entire treasure.

The pirates' first priority is to stay alive: they will act in such a way as to avoid death. If they can stay alive, they want to get as many coins as possible. Finally, they are a blood-thirsty bunch, if a pirate would get the same number of coins if she voted for or against a proposal, she will vote against so that the pirate who proposed the plan will be thrown overboard.

Assuming that all 5 pirates are intelligent (and aware that all the other pirates are just as aware, intelligent, and bloodthirsty), what will happen? Your solution should indicate which pirates die, and how many coins each of the remaining pirates receives.

10. Feedback

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment?
- Which problem did you spend the most time on?
- Which problem did you find to be the most confusing?
- Any other feedback for us?