Section 02: Digital Logic and Equivalence Proofs

1. Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

(a) $\neg p \to (s \to r)$ vs. $s \to (p \vee r)$

(b) $p \leftrightarrow \neg p$ vs. F (Hint: recall the Biconditional rule $p \leftrightarrow r \equiv (p \to r) \wedge (r \to p)$)

2. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) $p \to r$ vs. $r \to p$

(b) $a \to (b \wedge c)$ vs. $(a \to b) \wedge c$

3. They mean the same thing

Prove the following claims using chains of elementary equivalences, as shown in lecture:

$$\neg(\neg q \vee r) \equiv \neg(\neg q) \wedge \neg r \tag{1}$$
$$\neg(\neg q) \wedge \neg r \equiv q \wedge \neg r \tag{2}$$
$$q \wedge \neg r \equiv \neg r \wedge q \tag{3}$$

Your friend says this means that $\neg(q \to r) \equiv \neg r \wedge q$. Is that true?

4. Equivalent Translations

Prove that the following English statements are equivalent.
(i) Unless it isn't raining or I don't have an umbrella, I buy a book.
(ii) It isn't raining or I don't have an umbrella or I buy a book.

5. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) $\neg p \vee (\neg q \vee (p \wedge q))$

(b) $\neg(p \vee (q \wedge p))$

6. Properties of XOR

Like $\wedge$ and $\vee$, the $\oplus$ operator (exclusive or) has many interesting properties. For example, it is easy to verify with a truth table that $\oplus$ is also associative. In this problem, we will prove some additional properties of $\oplus$.

For this problem only, you may also use the equivalence

$$p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$$

which you may cite as "Definition of $\oplus$". This equivalence allows you to translate $\oplus$ into an expression involving only $\wedge$, $\vee$, and $\neg$, so that the standard equivalences can then be applied.

Prove the following claims using chains of elementary equivalences, as shown in lecture:

(a) $p \oplus q \equiv q \oplus p$ (Commutativity)

(b) $p \oplus p \equiv \mathsf{F}$ and $p \oplus \neg p \equiv \mathsf{T}$

(c) $p \oplus \mathsf{F} \equiv p$ and $p \oplus \mathsf{T} \equiv \neg p$

(d) $(\neg p) \oplus q \equiv \neg(p \oplus q) \equiv p \oplus (\neg q)$. I.e., negating one of the inputs negates the overall expression.