Section 02: Solutions

1. Equivalences

Prove that each of the following pairs of propositional formulas are equivalent using propositional equivalences.

(a) $\neg p \to (s \to r)$ vs. $s \to (p \vee r)$ **Solution:**

$$
\begin{aligned}
\neg p \to (s \to r) &\equiv \neg\neg p \vee (s \to r) && \text{Law of Implication} \\
&\equiv p \vee (s \to r) && \text{Double Negation} \\
&\equiv p \vee (\neg s \vee r) && \text{Law of Implication} \\
&\equiv (p \vee \neg s) \vee r && \text{Associativity} \\
&\equiv (\neg s \vee p) \vee r && \text{Commutativity} \\
&\equiv \neg s \vee (p \vee r) && \text{Associativity} \\
&\equiv s \to (p \vee r) && \text{Law of Implication}
\end{aligned}
$$

(b) $p \leftrightarrow \neg p$ vs. F (Hint: recall the Biconditional rule $p \leftrightarrow r \equiv (p \to r) \wedge (r \to p)$) **Solution:**

$$
\begin{aligned}
p \leftrightarrow \neg p &\equiv (p \to \neg p) \wedge (\neg p \to p) && \text{Biconditional} \\
&\equiv (\neg p \vee \neg p) \wedge (\neg\neg p \vee p) && \text{Law of Implication} \\
&\equiv (\neg p \vee \neg p) \wedge (p \vee p) && \text{Double Negation} \\
&\equiv \neg p \wedge p && \text{Idempotence} \\
&\equiv \mathsf{F} && \text{Negation}
\end{aligned}
$$

2. Non-equivalence

Prove that the following pairs of propositional formulae are not equivalent by finding inputs they differ on.

(a) $p \to r$ vs. $r \to p$ **Solution:**

When $p = \mathsf{T}$ and $r = \mathsf{F}$, then $p \to r \equiv \mathsf{F}$, but $r \to p \equiv \mathsf{T}$.

(b) $a \to (b \wedge c)$ vs. $(a \to b) \wedge c$ **Solution:**

When $a = \mathsf{F}$ and $c = \mathsf{F}$, then $a \to (b \wedge c) \equiv \mathsf{T}$ (by vacuous truth), but $(a \to b) \wedge c \equiv \mathsf{F}$ (because c is false).

3. They mean the same thing

Prove the following claims using chains of elementary equivalences, as shown in lecture:

$$\neg(\neg q \vee r) \equiv \neg(\neg q) \wedge \neg r \tag{1}$$

$$\neg(\neg q) \wedge \neg r \equiv q \wedge \neg r \tag{2}$$

$$q \wedge \neg r \equiv \neg r \wedge q \tag{3}$$

Your friend says this means that $\neg(q \to r) \equiv \neg r \wedge q$. Is that true?

Solution:

$$\begin{aligned}
\neg(q \to r) &\equiv \neg(\neg q \vee r) && \text{Law of Implication}\\
&\equiv \neg(\neg q) \wedge \neg r && \text{De Morgan}\\
&\equiv q \wedge \neg r && \text{Double Negation}\\
&\equiv \neg r \wedge q && \text{Commutativity}
\end{aligned}$$

For any statements A, B, and C, if A and B agree on all possible truth assignments and B and C do too, then A and C agree on all possible truth assignments, so the above chain of equivalences shows that $\neg(q \to r) \equiv \neg r \wedge q$.

4. Equivalent Translations

Prove that the following English statements are equivalent.
(i) Unless it isn't raining or I don't have an umbrella, I buy a book.
(ii) It isn't raining or I don't have an umbrella or I buy a book. **Solution:**

$$a : \text{It is raining.}$$

$$b : \text{I have an umbrella.}$$

$$c : \text{I buy a book.}$$

When we say unless a, b, this suggests that as long as a is not true, b will be true. Then, we can rewrite (i) as follows:

$$\neg(\neg a \vee \neg b) \to c$$

With the same propositional variables, we can rewrite (ii) as:

$$\neg a \vee \neg b \vee c$$

If these two compound propositions are equivalent, then the English statements are equivalent. Starting with the left-hand side

$$\begin{aligned}
\neg(\neg a \vee \neg b) \to c &\equiv (\neg\neg a \wedge \neg\neg b) \to c && \text{De Morgan}\\
&\equiv (a \wedge b) \to c && \text{Double Negation}\\
&\equiv \neg(a \wedge b) \vee c && \text{Law of Implication}\\
&\equiv (\neg a \vee \neg b) \vee c && \text{De Morgan}
\end{aligned}$$

Therefore, we've shown that the two English statements are equivalent.

5. Boolean Algebra

For each of the following parts, write the logical expression using boolean algebra operators. Then, simplify it using axioms and theorems of boolean algebra.

(a) $\neg p \vee (\neg q \vee (p \wedge q))$ **Solution:**

First, we replace $\neg, \vee$, and $\wedge$. This gives us $p' + q' + pq$; note that the parentheses are not necessary in boolean algebra, because the operations are all commutative and associative. We can use DeMorgan's laws to get the slightly simpler $(pq)' + pq$. Then, we can use commutativity to get $pq + (pq)'$ and complementarity to get 1. (Note that this is another way of saying the formula is a tautology.)

(b) $\neg(p \vee (q \wedge p))$ **Solution:**

First, we replace $\neg, \vee$, and $\wedge$ with their corresponding boolean operators, giving us $(p + (qp))'$. Applying DeMorgan's laws once gives us $p'(qp)'$, and a second time gives us $p'(q' + p')$, which is $p'(p' + q')$ by commutativity. By absorbtion, this is simply p'.

6. Properties of XOR

Like $\wedge$ and $\vee$, the $\oplus$ operator (exclusive or) has many interesting properties. For example, it is easy to verify with a truth table that $\oplus$ is also associative. In this problem, we will prove some additional properties of $\oplus$.

For this problem only, you may also use the equivalence

$$p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$$

which you may cite as "Definition of $\oplus$". This equivalence allows you to translate $\oplus$ into an expression involving only $\wedge$, $\vee$, and $\neg$, so that the standard equivalences can then be applied.

Prove the following claims using chains of elementary equivalences, as shown in lecture:

(a) $p \oplus q \equiv q \oplus p$ (Commutativity)

Solution:

$$\begin{aligned}
p \oplus q &\equiv (p \wedge \neg q) \vee (\neg p \wedge q) && \text{Definition of } \oplus \\
&\equiv (\neg p \wedge q) \vee (p \wedge \neg q) && \text{Commutativity} \\
&\equiv (q \wedge \neg p) \vee (\neg q \wedge p) && \text{Commutativity} \\
&\equiv q \oplus p && \text{Definition of } \oplus
\end{aligned}$$

(b) $p \oplus p \equiv \mathsf{F}$ and $p \oplus \neg p \equiv \mathsf{T}$

Solution:

$$\begin{aligned}
p \oplus p &\equiv (p \wedge \neg p) \vee (\neg p \wedge p) && \text{Definition of } \oplus \\
&\equiv (p \wedge \neg p) \vee (p \wedge \neg p) && \text{Commutativity} \\
&\equiv (p \wedge \neg p) && \text{Idempotency} \\
&\equiv \mathsf{F} && \text{Negation}
\end{aligned}$$

$$\begin{aligned}
p \oplus \neg p &\equiv (p \wedge \neg\neg p) \vee (\neg p \vee \neg p) && \text{Definition of } \oplus \\
&\equiv (p \wedge p) \vee (\neg p \vee \neg p) && \text{Double Negation} \\
&\equiv p \vee \neg p && \text{Idempotency} \\
&\equiv \mathsf{T} && \text{Negation}
\end{aligned}$$

(c) $p \oplus \mathsf{F} \equiv p$ and $p \oplus \mathsf{T} \equiv \neg p$

Solution:

$$\begin{aligned}
p \oplus \mathsf{F} &\equiv (p \wedge \neg \mathsf{F}) \vee (\neg p \wedge \mathsf{F}) && \text{Definition of } \oplus \\
&\equiv (p \wedge (\neg \mathsf{F} \vee \mathsf{F})) \vee (\neg p \wedge \mathsf{F}) && \text{Identity} \\
&\equiv (p \wedge (\mathsf{F} \vee \neg \mathsf{F})) \vee (\neg p \wedge \mathsf{F}) && \text{Commutativity} \\
&\equiv (p \wedge \mathsf{T}) \vee (\neg p \wedge \mathsf{F}) && \text{Negation} \\
&\equiv p \vee (\neg p \wedge \mathsf{F}) && \text{Identity} \\
&\equiv p \vee \mathsf{F} && \text{Domination} \\
&\equiv p && \text{Identity}
\end{aligned}$$

$$\begin{aligned}
p \oplus \mathsf{T} &\equiv (p \wedge \neg \mathsf{T}) \vee (\neg p \wedge \mathsf{T}) && \text{Definition of } \oplus \\
&\equiv (p \wedge \neg \mathsf{T}) \vee \neg p && \text{Identity} \\
&\equiv (\neg\neg p \wedge \neg \mathsf{T}) \vee \neg p && \text{Double Negation} \\
&\equiv \neg(\neg p \vee \mathsf{T}) \vee \neg p && \text{De Morgan} \\
&\equiv \neg \mathsf{T} \vee \neg p && \text{Domination} \\
&\equiv \neg(\mathsf{T} \wedge p) && \text{De Morgan} \\
&\equiv \neg(p \wedge \mathsf{T}) && \text{Commutativity} \\
&\equiv \neg p && \text{Identity}
\end{aligned}$$

(d) $(\neg p) \oplus q \equiv \neg(p \oplus q) \equiv p \oplus (\neg q)$. I.e., negating one of the inputs negates the overall expression.

Solution:

$$
\begin{aligned}
\neg(p \oplus q) &\equiv \neg((p \wedge \neg q) \vee (\neg p \wedge q)) && \text{Definition of } \oplus \\
&\equiv \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) && \text{De Morgan} \\
&\equiv (\neg p \vee \neg\neg q) \wedge (\neg\neg p \vee \neg q) && \text{De Morgan} \\
&\equiv (\neg p \vee q) \wedge (p \vee \neg q) && \text{Double Negation} \\
&\equiv ((\neg p \vee q) \wedge p) \vee ((\neg p \vee q) \wedge \neg q) && \text{Distributivity} \\
&\equiv (p \wedge (\neg p \vee q)) \vee (\neg q \wedge (\neg p \vee q)) && \text{Commutativity} \\
&\equiv ((p \wedge \neg p) \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) && \text{Distributivity} \\
&\equiv ((p \wedge \neg p) \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee (q \wedge \neg q)) && \text{Commutativity} \\
&\equiv (\mathsf{F} \vee (p \wedge q)) \vee ((\neg q \wedge \neg p) \vee \mathsf{F}) && \text{Negation} \\
&\equiv ((p \wedge q) \vee \mathsf{F}) \vee ((\neg q \wedge \neg p) \vee \mathsf{F}) && \text{Commutativity} \\
&\equiv (p \wedge q) \vee (\neg q \wedge \neg p) && \text{Negation} \\
&\equiv (\neg\neg p \wedge q) \vee (\neg q \wedge \neg p) && \text{Double Negation} \\
&\equiv (\neg\neg p \wedge q) \vee (\neg p \wedge \neg q) && \text{Commutativity} \\
&\equiv (\neg p \wedge \neg q) \vee (\neg\neg p \wedge q) && \text{Commutativity} \\
&\equiv \neg p \oplus q && \text{Definition of } \oplus
\end{aligned}
$$

The second equivalence $\neg(p \oplus q) \equiv p \oplus (\neg q)$ follows from the first and Commutativity (part a).