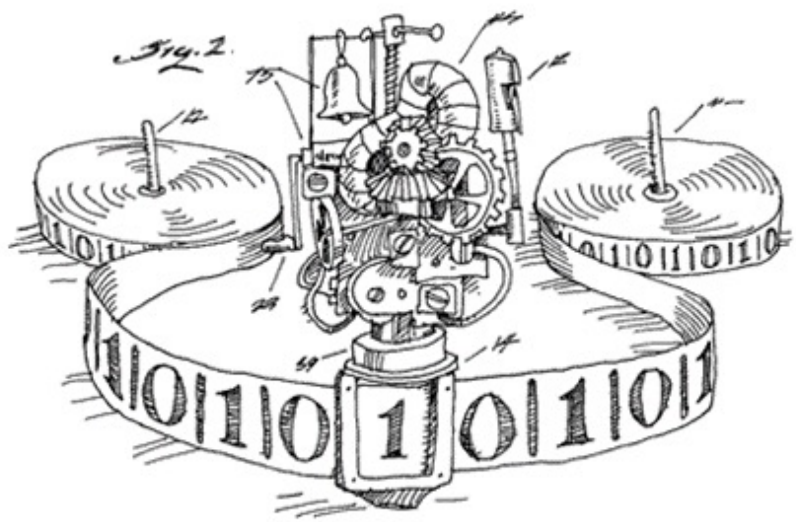
CSE 311: Foundations of Computing

Lecture 29: Reductions and Turing Machines



Final exam Monday, Review session Sunday

- **Monday** at either **2:30-4:20 (B)** or **4:30-6:20 (A)**
  - **CSE2 G20**
  - Bring your **UW ID**

- **Comprehensive:** Full probs only on topics that were covered in homework. May have small probs on other topics.
  - May includes pre-midterm topics, e.g., formal proofs.
  - Reference sheets will be included.

- **Review session: *Sunday* 1-3 pm in CSE2 G20**
  - **Bring your questions !!**

Final exam

- ? problems

Final exam

- 9 problems
- Large problems on:
  - DFA/RE/CFG design
  - DFA/NFA algorithms (except NFA to RE)
  - Irregularity
  - Strong & Structural Induction
  - English and Formal proofs about numbers/sets/relations/etc.
- Small problems on anything else

- 12 minutes per problem
  - write quickly
  - focus on the overall structure of the solution

Review: Countability vs Uncountability

- **To prove a set** A **countable, you must show**
  - **There exists a listing** $x_1, x_2, x_3, \ldots$ **such that every element of** A **is in the list.**

- **To prove a set** B **uncountable, you must show**
  - **For *every* listing** $x_1, x_2, x_3, \ldots$ **there exists some element in** B **that is not in the list.**
  - **The diagonalization proof shows how to describe a missing element** d **in** B **based on the listing** $x_1, x_2, x_3, \ldots$ **.**
    ***Important:* the proof produces a** d **no matter what the listing is.**

Last time: Undecidability of the Halting Problem

`CODE(P)` means "the code of the program **P**"

The Halting Problem

Given: - CODE(**P**) for any program **P**
- input **x**

Output: **true** if **P** halts on input **x**
false if **P** does not halt on input **x**

Theorem [Turing]: **There is no program that solves the Halting Problem**

Proof: By contradiction.

Assume that a program **H** solving the Halting program does exist. Then program **D** must exist

Does D(CODE(D)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); // don't halt
    } else {
        return;       // halt
    }
}
```

H solves the halting problem implies that

H(CODE(D),x) is **true** iff D(x) halts, H(CODE(D) ... ot

Suppose that D(CODE(D)) **halts**.

Then, by definition of H it mus ...

H(CODE(D), CO ...

Which by the defi ... D(CODE(D)) **doesn't halt**

Suppose th ... **doesn't halt**.

Th ... of H it must be that

... ODE(D), CODE(D)) is **false**

Whi ... by the definition of D means D(CODE(D)) **halts**

The ONLY assumption was the program H exists so that assumption must have been false.

Contradiction!

Where did the idea for creating D come from?

```
public static void D(s) {
    if (H(s,s) == true) {
        while (true);  // don't halt
    } else {
        return;        // halt
    }
}
```

D halts on input code(P) iff H(code(P),code(P)) outputs false
iff P doesn't halt on input code(P)

Connection to diagonalization

Write <**P**> for CODE(**P**)

Some possible inputs **x**

$<P_1>$ $<P_2>$ $<P_3>$ $<P_4>$ $<P_5>$ $<P_6>$ ….

All programs **P**

P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
.
.

This listing of all programs really does exist since the set of all Java programs is countable

The goal of this "diagonal" argument is not to show that the listing is incomplete but rather to show that a "flipped" diagonal element is not in the listing

Connection to diagonalization

Write <**P**> for CODE(**P**)

Some possible inputs **x**

All programs **P**

	$\langle P_1 \rangle$	$\langle P_2 \rangle$	$\langle P_3 \rangle$	$\langle P_4 \rangle$	$\langle P_5 \rangle$	$\langle P_6 \rangle$	....					
P_1	0	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1	0	1	0	...
P_9	.	.	. .	.	.	.	.	.	.	.	.	
.	.	.	. .	.	.	.	.	.	.	.	.	
.												

(P,x) entry is **1** if program **P** halts on input **x**
and **0** if it runs forever

Connection to diagonalization

Write <P> for CODE(P)

Some possible inputs **x**

All programs **P**

	$<P_1>$	$<P_2>$	$<P_3>$	$<P_4>$	$<P_5>$	$<P_6>$	....					
P_1	0^{1}	1	1	0	1							
P_2	1	1^{0}	0	1	0							
P_3	1	0	1^{0}	0	0							
P_4	0	1	1	0^{1}	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1^{0}	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1^{0}	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0^{1}	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1^{0}	0	1	0	...
P_9	. . . .											
.	. . . .											
.												

Want behavior of program ***D*** to be like the flipped diagonal, so it can't be in the list of all programs.

(P,x) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Where did the idea for creating D come from?

```
public static void D(s) {
    if (H(s,s) == true) {
        while (true);  // don't halt
    } else {
        return;        // halt
    }
}
```

D halts on input code(P) iff H(code(P),code(P)) outputs false
iff P doesn't halt on input code(P)

Therefore, for any program P, D differs from P on input code(P)

The Halting Problem isn't the only hard problem

- Can use the fact that the Halting Problem is undecidable to show that other problems are undecidable

General method (a "reduction"):

Prove that, if there were a program deciding B, then there would be a program deciding the Halting Problem.

"B decidable $\rightarrow$ Halting Problem decidable"

Contrapositive:

"Halting Problem undecidable $\rightarrow$ B undecidable"

Therefore, B is undecidable

A CSE 142 assignment

Students should write a Java program that:

- Prints “Hello” to the console
- Eventually exits

GradeIt, PracticeIt, etc. need to grade these

How do we write that grading program?

WE CAN’T: THIS IS IMPOSSIBLE!

Another undecidable problem

- **CSE 142 Grading problem:**
  - Input: CODE(Q)
  - Output:

    True if **Q** outputs "HELLO" and exits

    False if **Q** does not do that

- **Theorem:** The CSE 142 Grading is undecidable.
- Proof idea: Show that, if there is a program T to decide CSE 142 grading, then there is a program H to decide the Halting Problem for code(P) and input x.

Another undecidable problem

Theorem: The CSE 142 Grading is undecidable.

Proof: Suppose there is a program T that decide CSE 142 grading problem. Then, there is a program H to decide the Halting Problem for code(P) and input x by

- transform P (with input x) into the following program Q

Another undecidable problem

```
public class Q {
  private static String x = "...";

  public static void main(String[] args) {
    PrintStream out = System.out;
    System.setOut(new PrintStream(
        new WriterOutputStream(new StringWriter()));
    System.setIn(new ReaderInputStream(new StringReader(x)));

    P.main(args);

    out.println("HELLO");
  }
}

class P {
  public static void main(String[] args) { ... }
  ...
}
```

Another undecidable problem

Theorem: The CSE 142 Grading is undecidable.

Proof: Suppose there is a program T that decide CSE 142 grading problem. Then, there is a program H to decide the Halting Problem for code(P) and input x by

- transform P (with input x) into the following program Q
- run T on code(Q)
  - if it returns true, then P halted

    must halt in order to print “HELLO”
  - if it returns false, then P did not halt

    program Q can’t output anything other than “HELLO”

More Reductions

- Can use undecidability of these problems to show that other problems are undecidable.

- For instance:

$\mathrm{EQUIV}(P, Q)$:	**True**	if $P(x)$ and $Q(x)$ have the same behavior for every input x
	False	otherwise

Rice's theorem

Not *every* problem on programs is undecidable!

Which of these is decidable?

- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x
  after less than 100 steps
  false otherwise

- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x
  after more than 100 steps
  false otherwise

Rice's Theorem:

Any "non-trivial" property of the **input-output behavior** of Java programs is undecidable.

Rice's theorem

Not *every* problem on programs is undecidable!

Which of these is decidable?

- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x after less than 100 steps
  false otherwise

- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x after more than 100 steps
  false otherwise

Rice's Theorem (a.k.a. Compilers ARE DIFFICULT):
Any "non-trivial" property of the **input-output behavior** of Java programs is undecidable.

CFGs are complicated

We know can answer almost any question about REs

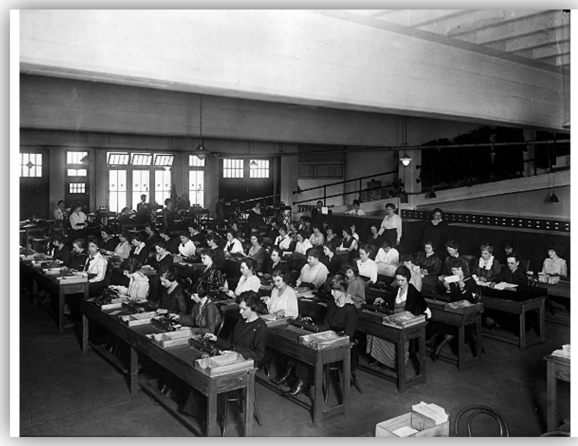
- Do two REs / DFAs recognize the same language?

But many problems about CFGs are undecidable!

- **Do two CFGs generate the same language?**
- **Is there any string that two CFGs both generate?**
  - **more general: "CFG intersection" problem**
- **Does a CFG generate *every* string?**

Computers and algorithms

- Does Java (or any programming language) cover all possible computation? Every possible algorithm?
- There was a time when computers were people who did calculations on sheets paper to solve computational problems



- Computers as we known them arose from trying to understand everything these people could do.

Before Java

1930's:

How can we formalize what algorithms are possible?

- **Turing machines** (Turing, Post)
  - basis of modern computers
- **Lambda Calculus** (Church)
  - basis for functional programming, LISP
- **μ-recursive functions** (Kleene)
  - alternative functional programming basis

Turing machines

Church-Turing Thesis:

Any reasonable model of computation that includes all possible algorithms is equivalent in power to a Turing machine

Evidence

- Huge numbers of models based on radically different ideas turned out to be equivalent to TMs
- TM can simulate the physics of any machine that we could build (even quantum computers)

Turing machines

- **Finite Control**
  - **Brain/CPU** that has only a **finite # of possible "states of mind"**
- **Recording medium**
  - **An unlimited supply of** blank **"scratch paper"** on which to **write & read symbols,** each chosen from a **finite set of possibilities**
  - **Input** also supplied **on** the **scratch paper**
- **Focus of attention**
  - **Finite control** can only **focus on** a **small portion of the recording medium at once**
  - **Focus of attention** can only **shift a small amount at a time**

Turing machines

- Recording medium
  - An **infinite** read/write "**tape**" marked off into **cells**
  - Each **cell** can store **one symbol or** be "**blank**"
  - **Tape** is **initially** all **blank except** a few **cells** of the tape **containing the input string**
  - **Read/write head** can **scan one cell** of the tape - **starts on input**

- In each step, **a Turing machine**
  1. **Reads** the **currently scanned cell**
  2. **Based on current state and scanned symbol**
     1. **Overwrites symbol in scanned cell**
     2. **Moves read/write head left or right one cell**
     3. **Changes to a new state**

- **Each Turing Machine is specified by its finite set of rules**

Turing machines

	_	0	1
s_1	(1, L, s_3)	(1, L, s_4)	(0, R, s_2)
s_2	(0, R, s_1)	(1, R, s_1)	(0, R, s_1)
s_3			
s_4			

_	_	1	1	0	1	1	_	_

UW CSE’s Steam-Powered Turing Machine

Original in Sieg Hall stairwell

Turing machines

Ideal Java/C programs:

- Just like the Java/C you're used to programming with, except you never run out of memory
    - no OutOfMemoryError

Equivalent to Turing machines but easier to program:

- Turing machine definition is useful for breaking computation down into simplest steps
- We only care about high level so we use programs

Turing's big idea part 1: Machines as data

Original Turing machine definition:

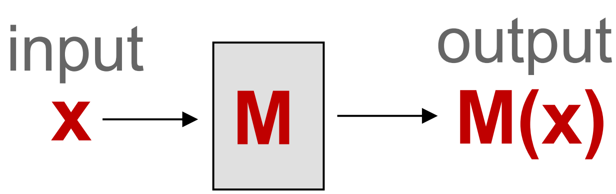
- A different "machine" **M** for each task
- Each machine **M** is defined by a finite set of possible operations on finite set of symbols
- So… **M** has a finite description as a sequence of symbols, its "code", which we denote **<M>**

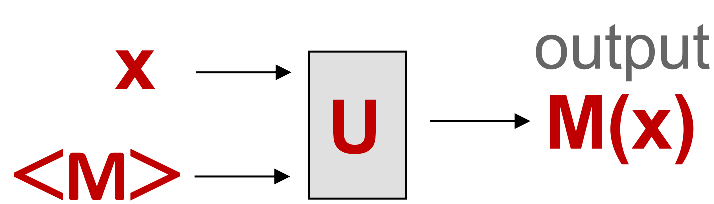
You already are used to this idea with the notion of the program code, but this was a new idea in Turing's time.

Turing's big idea part 2: A Universal TM

- **A Turing machine interpreter U**
  - On input <M> and its input x,
    U outputs the same thing as M does on input x
  - At each step it decodes which operation M would have performed and simulates it.
- **One Turing machine is enough**
  - Basis for modern stored-program computer

    Von Neumann studied Turing's UTM design





Takeaway from undecidability

- You can't rely on the idea of improved compilers and programming languages to eliminate all programming errors
  - truly safe languages can't possibly do general computation
- Document your code
  - there is no way you can expect someone else to figure out what your program does with just your code; since in general it is provably impossible to do this!

What's next? ...after the final exam...

- **Foundations II** (312)
  - Fundamentals of counting, discrete probability, applications of randomness to computing, statistical algorithms and analysis
  - Ideas critical for machine learning, algorithms

- **Data Abstractions** (332)
  - Data structures, a few key algorithms, parallelism
  - Brings programming and theory together
  - Makes heavy use of induction and recursive defns

More Complexity Theory

Not just what can be computed at all...

How about what can be computed *efficiently*?

A rich, interesting, and important topic.

See CSE 431 for much more on that!

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