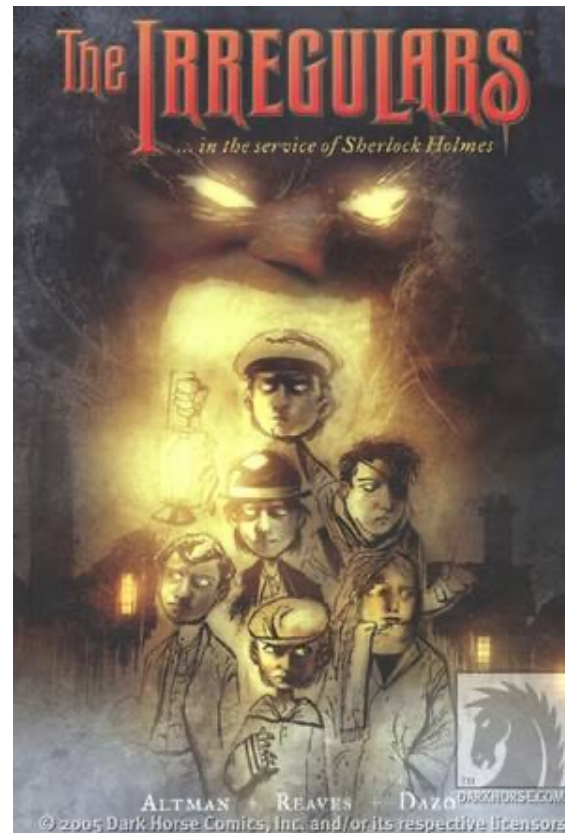# CSE 311: Foundations of Computing

Lecture 26: Languages vs Representations:
Limitations of Finite Automata and Regular Expressions

Last time: Algorithms for Regular Languages
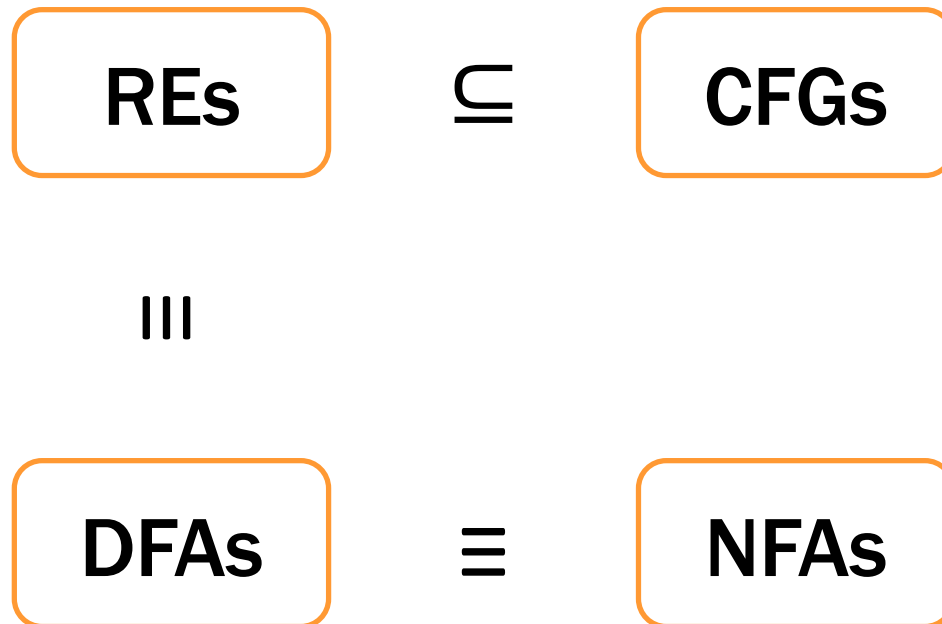
We have seen algorithms for

- RE to NFA
- NFA to DFA
- DFA/NFA to RE   (not tested)
- DFA minimization

Practice three of these in HW.

(May also be on the final.)

The story so far...

REs $\subseteq$ CFGs

$\equiv$

DFAs $\equiv$ NFAs

Languages represented by DFA, NFAs, or regular expressions are called **Regular Languages**

Regular expressions ≡ NFAs ≡ DFAs

We have shown how to build an optimal DFA for every regular expression

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Thus, we could now implement a RegExp library

- most RegExp libraries actually simulate the NFA
- (even better: one can combine the two approaches: apply DFA minimization lazily while simulating the NFA)

Application of FSMs: Pattern matching

- **Given**

  - a string s of n characters

  - a pattern p of m characters

  - usually $m \ll n$

- **Find**

  - all occurrences of the pattern p in the string s

- Obvious algorithm:

  - try to see if p matches at each of the positions in s stop at a failed match and try matching at the next position: $O(mn)$ running time.

Application of FSMs: Pattern Matching

- With DFAs can do this in $O(m + n)$ time.

- See Extra Credit problem on HW8 for some ideas of how to get to $O(m^2 + n)$.

Last time: NFA to DFA

NFA

DFA
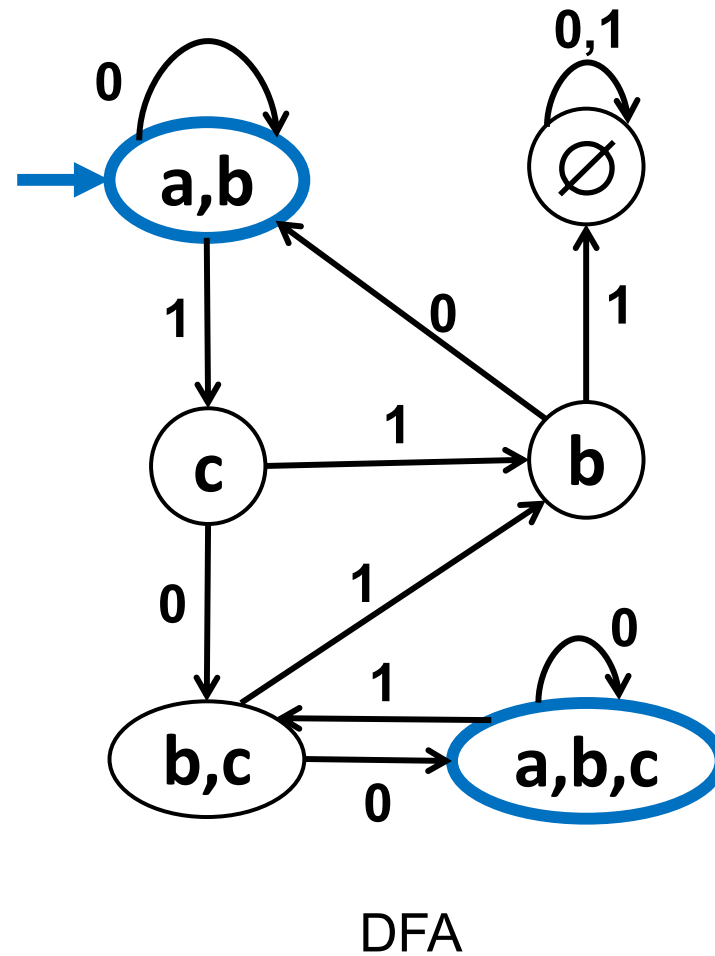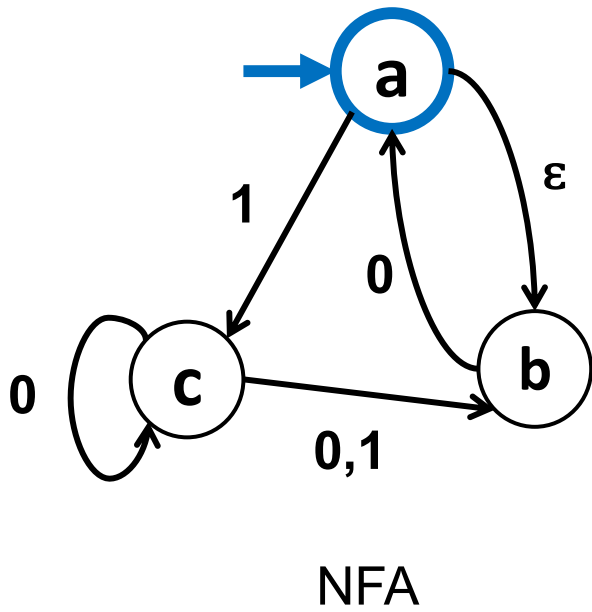
Exponential Blow-up in Simulating Nondeterminism

- **In general the DFA might need a state for every subset of states of the NFA**
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - We saw an example where roughly 2^n is necessary

    "Is the nth char from the end a 1?"

The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms

The story so far...

$$\text{REs} \subseteq \text{CFGs}$$

$$\equiv$$

$$\text{DFAs} \equiv \text{NFAs}$$
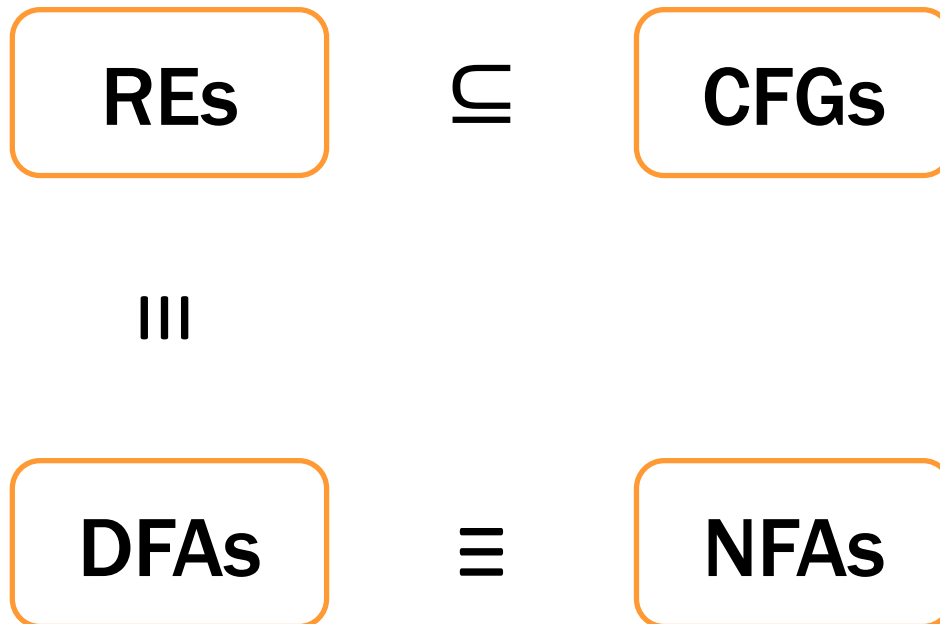
What languages have DFAs? CFGs?

All of them?

Languages and Representations!

All

Context-Free

Regular

0*

DFA
NFA
Regex

Finite

{001, 10, 12}

Languages and Representations!

All

Context-Free

Regular

0*

DFA
NFA
Regex

Finite

{001, 10, 12}

Warmup:
All finite
languages
are regular.
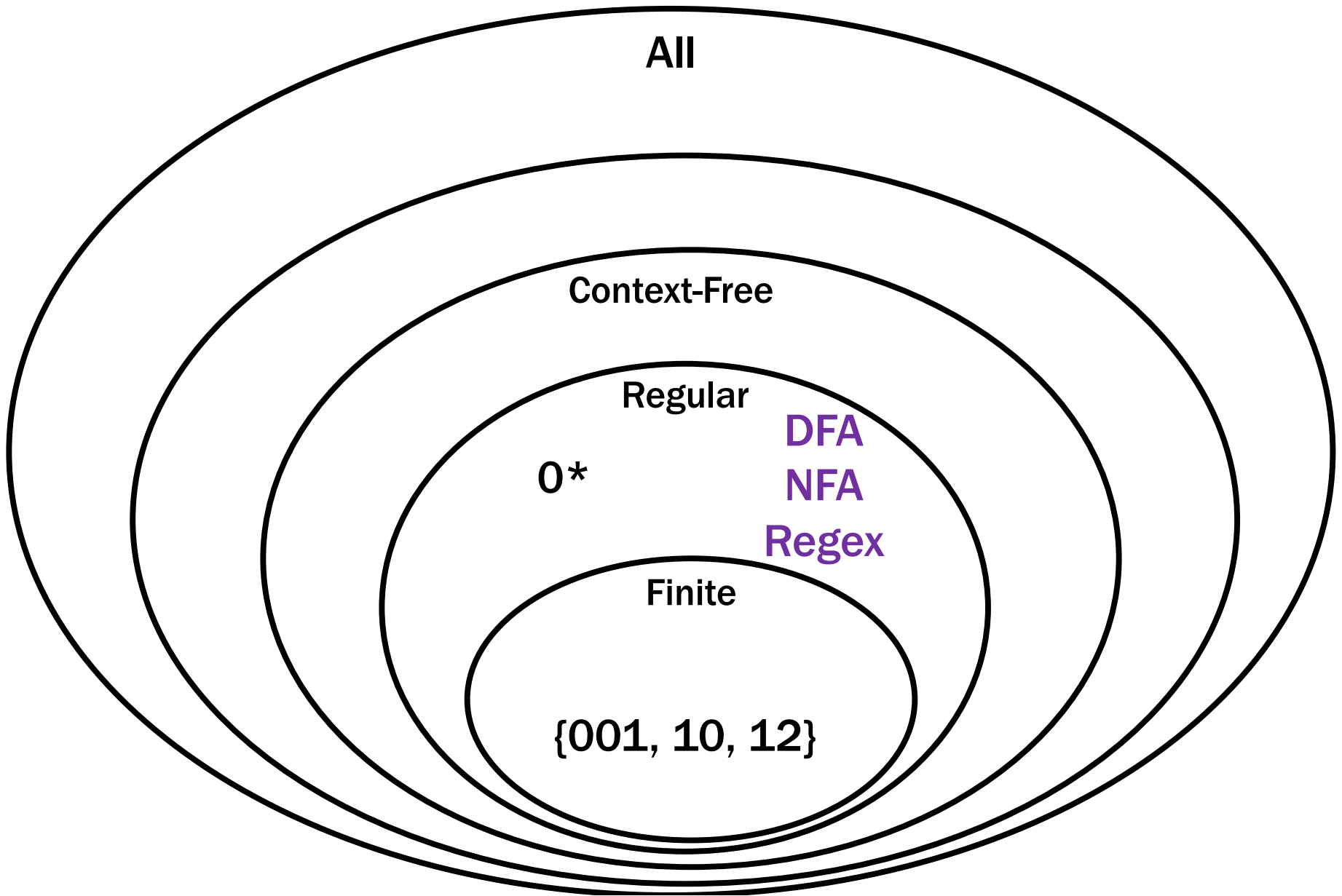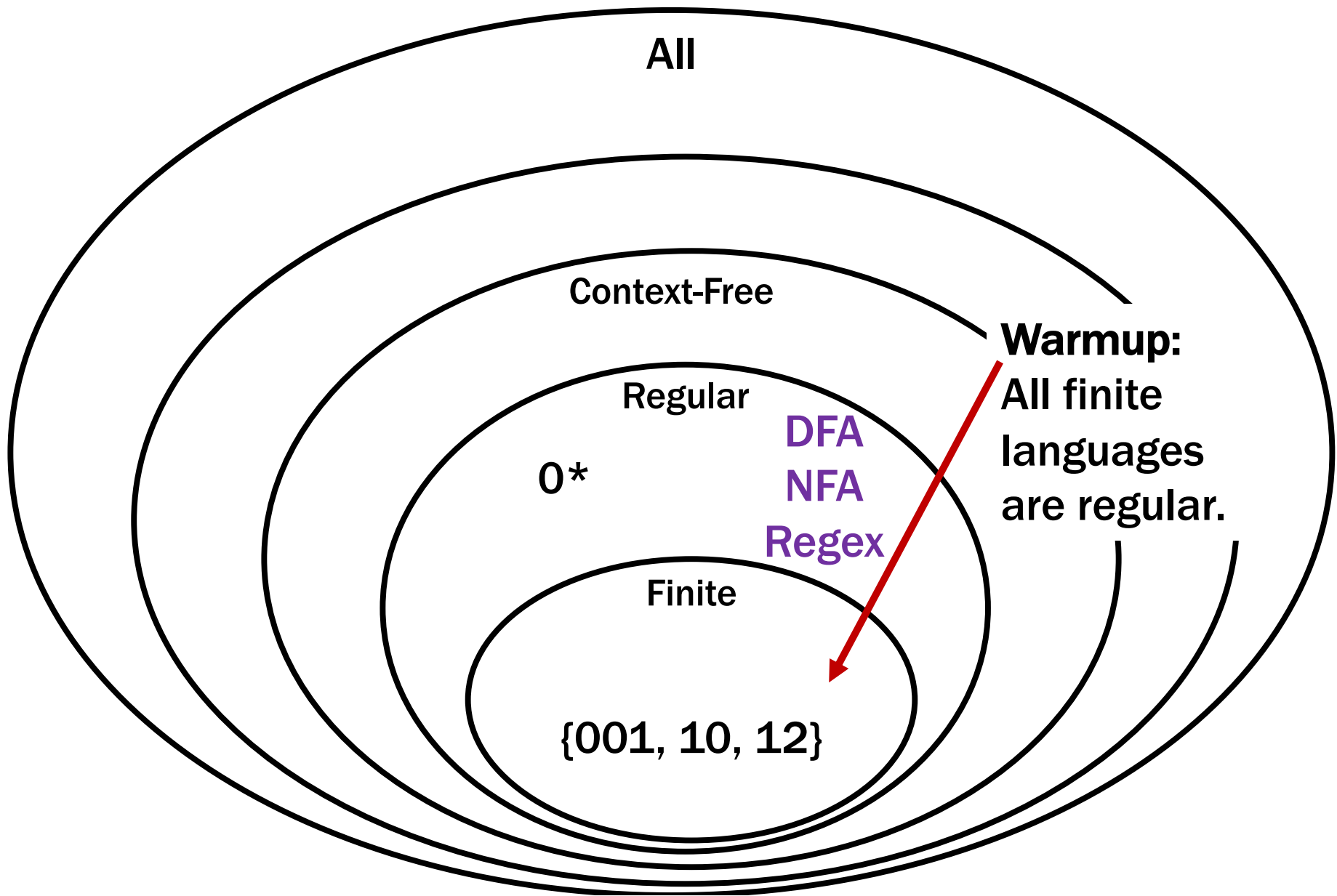
DFAs Recognize Any Finite Language

DFAs Recognize Any Finite Language

Construct a DFA for each string in the language.

Then, put them together using the union construction.

Languages and Machines!
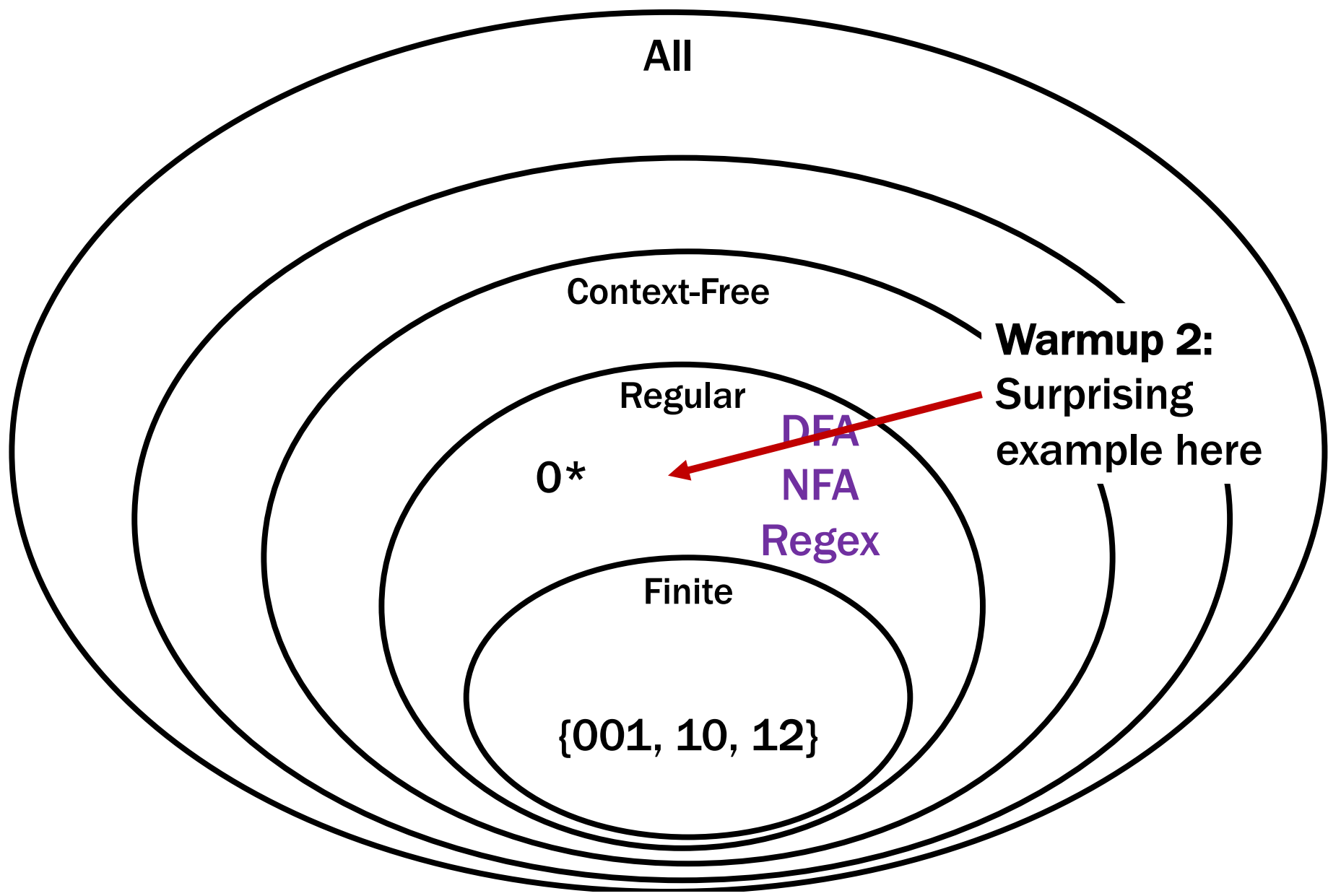

All

Context-Free

Regular

0*

DFA
NFA
Regex

Finite

{001, 10, 12}

Warmup 2: Surprising example here

An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

L is infinite.

  0, 00, 000, ...

L is regular. How could this be?

That seems to require comparing counts...

- – easy for a CFG

- – but seems hard for DFAs!

An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

L is infinite.

   0, 00, 000, …

L is regular. How could this be?   It is just the set of binary strings that are empty or begin and end with the same character!

Languages and Representations!

All

Context-Free

???

Regular

0*

DFA
NFA
Regex

Finite

{001, 10, 12}

Main Event: Prove there is a context-free language that isn't regular.
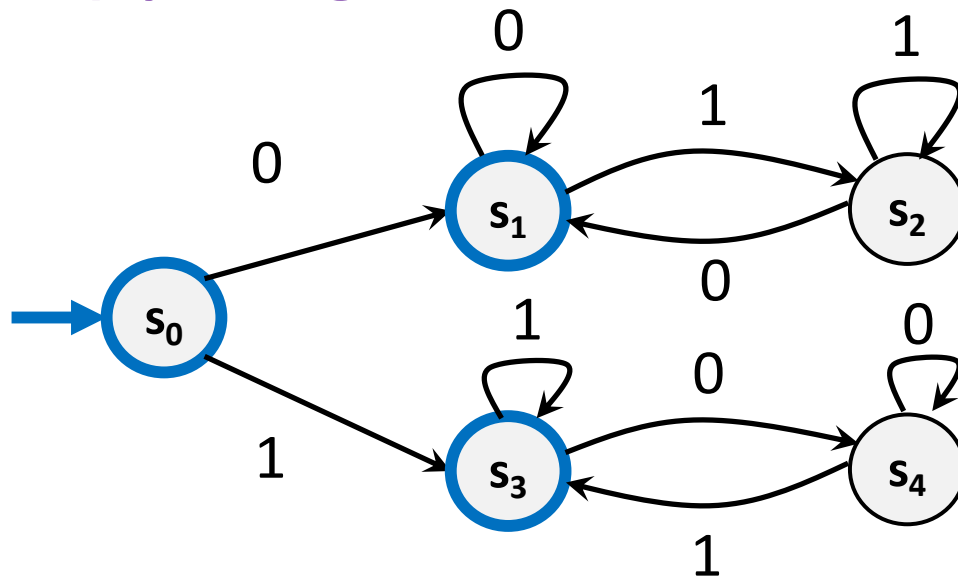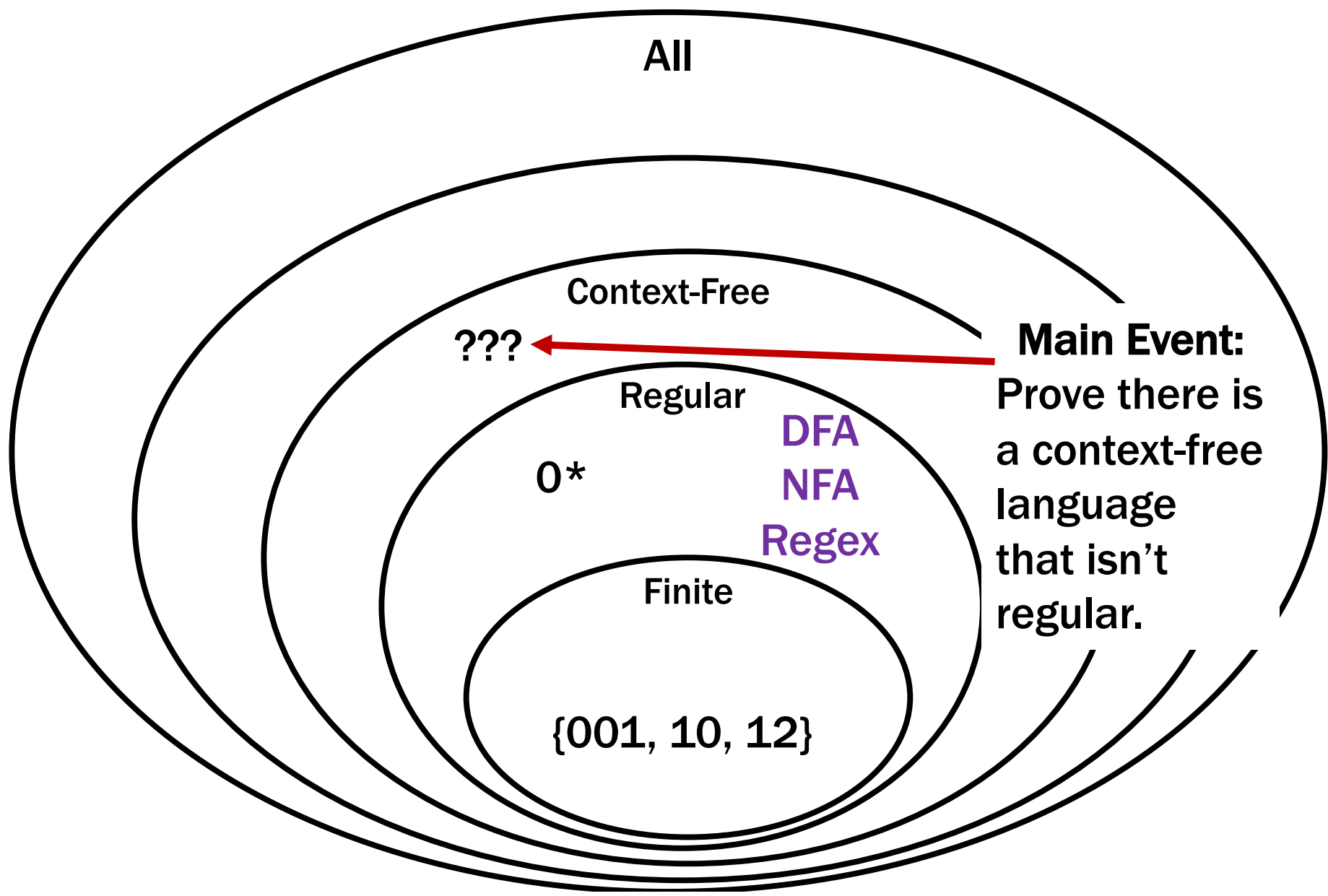
The language of "Binary Palindromes" is Context-Free

$$S \rightarrow \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

Is the language of "Binary Palindromes" Regular ?

Intuition (NOT A PROOF!):

Q: What would a DFA need to keep track of to decide?

A: It would need to keep track of the "first part" of the input in order to check the second part against it

...but there are an infinite # of possible first parts and we only have finitely many states.

Proof idea: any machine that does not remember the entire first half will be wrong for some inputs

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- – Assume (for contradiction) that
  some DFA (call it **M**) exists that recognizes **B**

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:
- – Assume (for contradiction) that
  some DFA (call it **M**) exists that recognizes **B**
- – Our goal is to show that **M** actually does not
  recognize **B**

How can a DFA fail to recognize **B**?
- – when it accepts or rejects a string it shouldn't.

B = {binary palindromes} can't be recognized by any DFA

___

The general proof strategy is:

- – Assume (for contradiction) that some DFA (call it **M**) exists that recognizes **B**
- – Our goal is to show that **M** actually does not recognize **B**, i.e., it accepts or rejects a string that it shouldn't
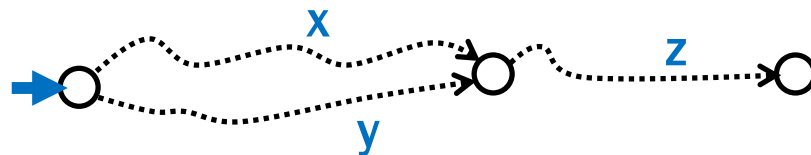
"**M** recognizes **B**" AND "**M** doesn't recognize **B**", which is a contradiction

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that
  some DFA (call it **M**) exists that recognizes **B**

- We want to show: **M** accepts or rejects a string it shouldn't.

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!

$$\textbf{M is correct} \ \ \textbf{iff} \ \ \forall z \in \Sigma^* \, (xz \in \textbf{B} \leftrightarrow yz \in \textbf{B})$$
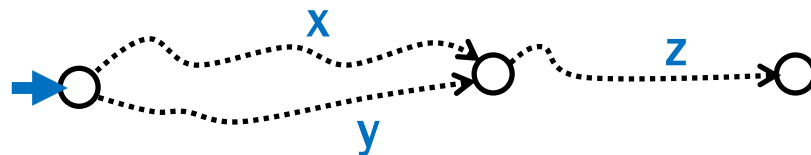
$$\textbf{M is incorrect} \ \ \textbf{iff} \ \ \exists z \in \Sigma^* \, (xz \in \textbf{B} \not\leftrightarrow yz \in \textbf{B})$$

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that
  some DFA (call it **M**) exists that recognizes **B**

- We want to show: **M** accepts or rejects a string it shouldn't.

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!

Key Idea 2: Our machine M has a finite number of states which means if we have *infinitely many* strings, two of them must collide!

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

- – Assume (for contradiction) that
  some DFA (call it **M**) exists that recognizes **B**

- – We want to show: **M** accepts or rejects a string it shouldn't.

We choose an **INFINITE** set **S** of "partial strings"
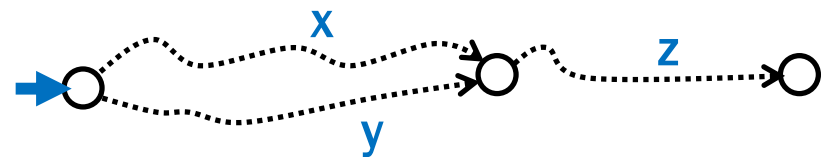(which we intend to complete later).

x,y   z

1_____
01_____
001_____
0001_____
00001_____
....

B = {binary palindromes} can't be recognized by any DFA

The general proof strategy is:

– Assume (for contradiction) that
some DFA (call it M) exists that recognizes B

– We want to show: M accepts or rejects a string it shouldn't.

We choose an **INFINITE** set S of "partial strings" (which we intend to complete later). It is critical that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.

```
x,y      z
    1_____
   01_____
  001_____
 0001_____
00001_____
..........
```
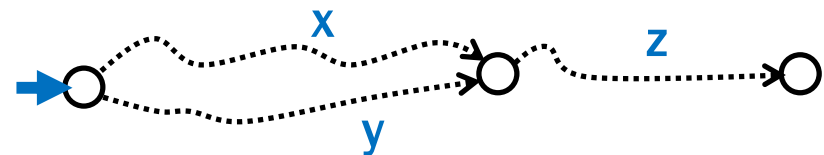
B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

Key Idea 2: Our machine has a finite number of states which means if we have infinitely many strings, two of them must collide!

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, **M**, recognizes **B**.

We show **M** accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, ...\} = \{0^n1 : n \geq 0\}$.

*Since there are finitely many states in **M** and infinitely many strings in S, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of **M**.*

SUPER IMPORTANT POINT: You do not get to choose what a and b are. Remember, we've just proven they exist...we have to take the ones we're given!

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, **M**, accepts **B**.

We show **M** accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, ...\} = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in **M** and infinitely many strings in **S**, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of **M**.

Now, consider appending 0^a to both strings.

Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!

B = {binary palindromes} can't be recognized by any DFA

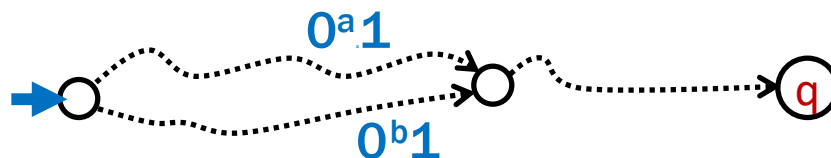Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, ...\} = \{0^n 1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, there exist strings $0^a 1 \in S$ and $0^b 1 \in S$ with $a \neq b$ that end in the same state of M.

Now, consider appending 0^a to both strings.

Then, since $0^a 1$ and $0^b 1$ end in the same state, $0^a 1 0^a$ and $0^b 1 0^a$ also end in the same state, call it q.

But then M makes a mistake: q needs to be an accept state since $0^a 1 0^a \in B$, but M would accept $0^b 1 0^a \notin B$ which is an error.

B = {binary palindromes} can't be recognized by any DFA
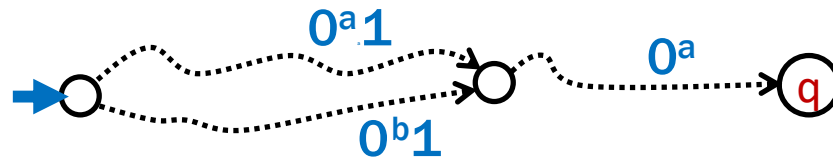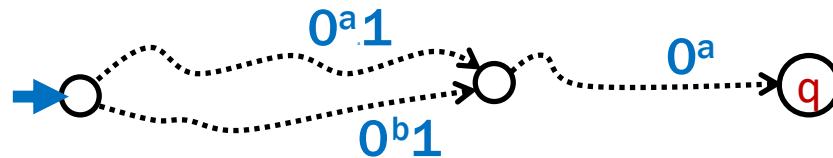
Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider $S = \{1, 01, 001, 0001, 00001, \ldots\} = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S, there exist strings $0^a1 \in S$ and $0^b1 \in S$ with $a \neq b$ that end in the same state of M.

Now, consider appending 0^a to both strings.

Then, since 0^a1 and 0^b1 end in the same state, 0^a10^a and 0^b10^a also end in the same state, call it q. But then M must make a mistake: q needs to be an accept state since $0^a10^a \in B$, but then M would accept $0^b10^a \notin B$ which is an error.

This is a contradiction since we assumed that M recognizes B. Thus, no DFA recognizes B.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."

2. Consider an **INFINITE** set S of "partial strings" (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.

3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for $s_a \neq s_b$ that end up at the same state of M."

4. Consider appending the (correct) completion t to each of the two strings.

5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both $s_a t$ and $s_b t$ end at the same state q of M. Since $s_a t \in L$ and $s_b t \notin L$, M does not recognize L."

6. "Thus, no DFA recognizes L."

Prove $A = \{0^n 1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let $S =$

Prove $A = \{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let $S = \{0^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in M.

Prove $A = \{0^n 1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let $S = \{0^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in M.

Consider appending 1^a to both strings.

Prove $A = \{0^n1^n : n \geq 0\}$ is not regular

Suppose for contradiction that some DFA, M, recognizes A.

Let $S = \{0^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, 0^a and 0^b for some $a \neq b$ that end in the same state in M.

Consider appending 1^a to both strings.

Note that $0^a1^a \in A$, **but** $0^b1^a \notin A$ since $a \neq b$. But they both end up in the same state of M, call it **q**. Since $0^a1^a \in A$, state **q** must be an accept state but then M would incorrectly accept $0^b1^a \notin A$ so M does not recognize A.

Thus, no DFA recognizes A.

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, accepts P.

Let S =

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let $S = \{ \,(^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, $(^a$ and $(^b$ for some $a \neq b$ that end in the same state in M.

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let S = { $($ ^{n} : n ≥ 0}. Since S is infinite and M has finitely many states, there must be two strings, $($ ^{a} and $($ ^{b} for some a ≠ b that end in the same state in M.

Consider appending $)$ ^{a} to both strings.

Prove P = {balanced parentheses} is not regular

Suppose for contradiction that some DFA, M, recognizes P.

Let $S = \{ (^n : n \geq 0\}$. Since S is infinite and M has finitely many states, there must be two strings, $(^a$ and $(^b$ for some $a \neq b$ that end in the same state in M.

Consider appending $)^a$ to both strings.

Note that $(^a)^a \in P$, **but** $(^b)^a \notin P$ since $a \neq b$. But they both end up in the same state of M, call it q. Since $(^a)^a \in P$, state q must be an accept state but then M would incorrectly accept $(^b)^a \notin P$ so M does not recognize P.

Thus, no DFA recognizes P.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA **M** recognizes **L**."

2. Consider an **INFINITE** set **S** of "partial strings" (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.

3. "Since **S** is infinite and **M** has finitely many states, there must be two strings s_a and s_b in **S** for $s_a \neq s_b$ that end up at the same state of **M**."

4. Consider appending the (correct) completion **t** to each of the two strings.

5. "Since s_a and s_b both end up at the same state of **M**, and we appended the same string **t**, both $s_a t$ and $s_b t$ end at the same state **q** of **M**. Since $s_a t \in$ **L** and $s_b t \notin$ **L**, **M** does not recognize **L**."

6. "Thus, no DFA recognizes **L**."

Fact: This method is optimal

- Suppose that for a language L, the set S is a *largest* set of "partial strings" with the property that, for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t$, $s_b t$ is in L but the other isn't.

- If S is infinite, then L is not regular

- If S is finite, then the minimal DFA for L has precisely |S| states, one reached by each member of S.

Fact: This method is optimal

- Suppose that for a language L, the set S is a *largest* set of "partial strings" with the property that, for every pair $s_a \neq s_b \in S$, there is some string t such that one of $s_a t$, $s_b t$ is in L but the other isn't.

- If S is infinite, then L is not regular

- If S is finite, then the minimal DFA for L has precisely |S| states, one reached by each member of S.

Corollary: Our minimization algorithm was correct.

  – we separated *exactly* those states for which some t would make one accept and another not accept

Important Notes

- It is not necessary for our strings **xz** with **x** $\in$ **L** to allow any string in the language
  - we only need to find a small "core" set of strings that must be distinguished by the machine

- It is **not true** that, if **L** is irregular and **L** $\subseteq$ **U**, then **U** is irregular!
  - we always have **L** $\subseteq$ **Σ*** and **Σ*** is regular!
  - our argument needs different answers: $xz \in$ **L** $\not\leftrightarrow$ $yz \in$ **L**
    for **Σ***, both strings are always in the language

> Do not claim in your proof that, because **L** $\subseteq$ **U**, **U** is also irregular