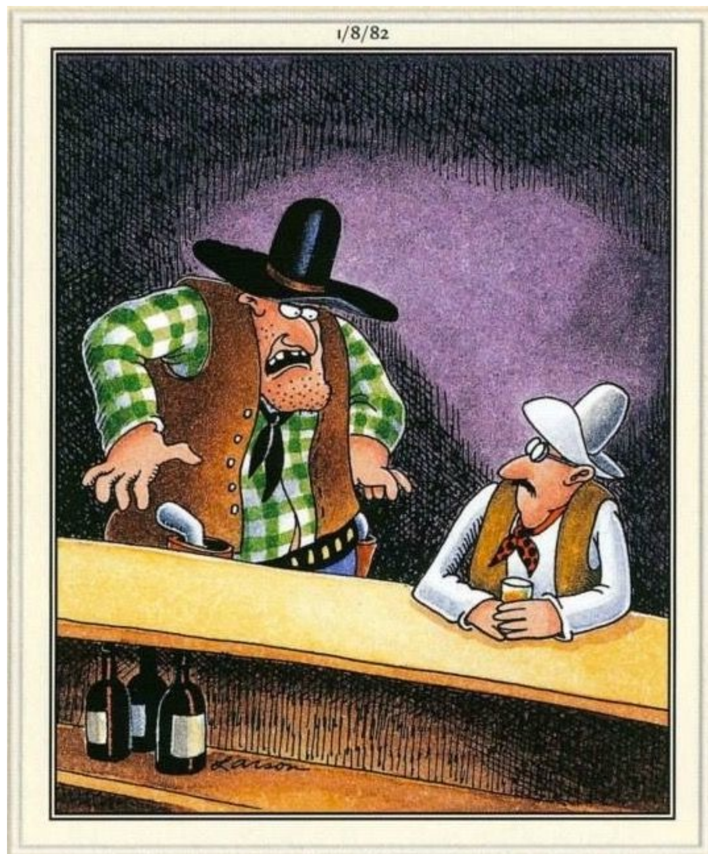
CSE 311: Foundations of Computing

Lecture 14: Modular Inverse, Exponentiation



"I *asked* you a question, buddy. ... What's the square root of 5,248?"

Last time: Useful GCD Facts

If a and b are positive integers, then

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

If a is a positive integer, $\gcd(a, 0) = a$.

Euclid's Algorithm

gcd(a, b) = gcd(b, a mod b)   gcd(a, 0) = a

```
int gcd(int a, int b){ /* Assumes: a >= b, b >= 0 */
    if (b == 0) {
        return a;
    } else {
        return gcd(b, a % b);
    }
}
```

Note: gcd(b, a) = gcd(a, b)

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) =

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) = gcd(126, 660 mod 126) = gcd(126, 30)
= gcd(30, 126 mod 30) = gcd(30, 6)
= gcd(6, 30 mod 6) = gcd(6, 0)
= 6

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

Equations with recursive calls:

gcd(660,126) = gcd(126, 660 mod 126) = gcd(126, 30)
= gcd(30, 126 mod 30) = gcd(30, 6)
= gcd(6, 30 mod 6) = gcd(6, 0)
= 6

Tableau form:

660 = 5 * 126 + 30
126 = 4 * 30 + 6
30 = 5 * 6 + 0

Famous Algorithmic Problems

- **Primality Testing**
  - Given an integer n, determine if n is prime
- **Factoring**
  - Given an integer n, find an integer d (with $1 < d < n$) that divides n
- **Greatest Common Divisor**
  - Given integers a and b, find the largest integer d that divides them both

Bézout's theorem

If a and b are positive integers, then there exist integers $\boldsymbol{s}$ and $\boldsymbol{t}$ such that

$$\gcd(a,b) = \boldsymbol{s}a + \boldsymbol{t}b.$$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 1 (Compute GCD & Keep Tableau Information):

a b b a mod b = r b r

$$\gcd(35, 27) = \gcd(27, 35 \bmod 27) = \gcd(27, 8)$$

a = q * b + r

$$35 = 1 * 27 + 8$$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 1 (Compute GCD & Keep Tableau Information):

$$\gcd(\overset{a}{35}, \overset{b}{27}) = \gcd(\overset{b}{27}, \overset{a \bmod b = r}{35 \bmod 27}) = \gcd(\overset{b}{27}, \overset{r}{8})$$

$$= \gcd(8, 27 \bmod 8) = \gcd(8, 3)$$

$$= \gcd(3, 8 \bmod 3) = \gcd(3, 2)$$

$$= \gcd(2, 3 \bmod 2) = \gcd(2, 1)$$

$$= \gcd(1, 2 \bmod 1) = \gcd(1, 0)$$

a = q * b + r

$35 = 1 * 27 + 8$

$27 = 3 * 8 + 3$

$8 = 2 * 3 + 2$

$3 = 1 * 2 + 1$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 2 (Solve the equations for r):

a = q * b + r
$35 = 1 * 27 + 8$
$27 = 3 * 8 + 3$
$8 = 2 * 3 + 2$
$3 = 1 * 2 + 1$

r = a – q * b

$8 = 35 - 1 * 27$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 2 (Solve the equations for r):

a = q * b + r	r = a – q * b
$35 = 1 * 27 + 8$	$8 = 35 - 1 * 27$
$27 = 3 * 8 + 3$	$3 = 27 - 3 * 8$
$8 = 2 * 3 + 2$	$2 = 8 - 2 * 3$
$3 = 1 * 2 + 1$	$1 = 3 - 1 * 2$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Plug in the def of 2

$$\begin{aligned} 1 &= 3 - 1 * (8 - 2 * 3) \\ &= 3 - 8 + 2 * 3 \\ &= (-1) * 8 + 3 * 3 \end{aligned}$$

Re-arrange into 3's and 8's

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Plug in the def of 2

$$1 = 3 - 1 * (8 - 2 * 3)$$
$$= 3 - 8 + 2 * 3$$
$$= (-1) * 8 + 3 * 3$$

Re-arrange into 3's and 8's

Plug in the def of 3

$$= (-1) * 8 + 3 * (27 - 3 * 8)$$
$$= (-1) * 8 + 3 * 27 + (-9) * 8$$
$$= 3 * 27 + (-10) * 8$$

Re-arrange into 8's and 27's

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Plug in the def of 2

$$1 = 3 - 1 * (8 - 2 * 3)$$
$$= 3 - 8 + 2 * 3$$
$$= (-1) * 8 + 3 * 3$$

Re-arrange into 3's and 8's

Plug in the def of 3

$$= (-1) * 8 + 3 * (27 - 3 * 8)$$
$$= (-1) * 8 + 3 * 27 + (-9) * 8$$
$$= 3 * 27 + (-10) * 8$$

Re-arrange into 8's and 27's

$$= 3 * 27 + (-10) * (35 - 1 * 27)$$
$$= 3 * 27 + (-10) * 35 + 10 * 27$$
$$= 13 * 27 + (-10) * 35$$

Re-arrange into 27's and 35's

Multiplicative inverse $\mathrm{mod}\ m$

Let $0 \leq a, b < m$. Then, b is the *multiplicative inverse of* a *(modulo* m*)* iff $ab \equiv_m 1$.

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

mod 7

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

mod 10

Multiplicative inverse $\bmod m$

Suppose $\gcd(a, m) = 1$

By Bézout's Theorem, there exist integers s and t such that $sa + tm = 1$.

s is the multiplicative inverse of a (modulo m):

$$1 = sa + tm \equiv_m sa$$

So… we can compute multiplicative inverses with the extended Euclidean algorithm

These inverses let us solve modular equations…

Example

Solve: $7x \equiv_{26} 1$

Example

Solve: $7x \equiv_{26} 1$

$$\gcd(26, 7) = \gcd(7, 5) = \gcd(5, 2) = \gcd(2, 1) = 1$$

Example

Solve: $7x \equiv_{26} 1$

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$26 = 3 * 7 + 5$$
$$7 = 1 * 5 + 2$$
$$5 = 2 * 2 + 1$$

Example

Solve: $7x \equiv_{26} 1$

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$26 = 3 * 7 + 5 \qquad 5 = 26 - 3 * 7$$

$$7 = 1 * 5 + 2 \qquad 2 = 7 - 1 * 5$$

$$5 = 2 * 2 + 1 \qquad 1 = 5 - 2 * 2$$

Example

Solve: $7x \equiv_{26} 1$

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$26 = 3 * 7 + 5 \qquad 5 = 26 - 3 * 7$$

$$7 = 1 * 5 + 2 \qquad 2 = 7 - 1 * 5$$

$$5 = 2 * 2 + 1 \qquad 1 = 5 - 2 * 2$$

$$\begin{aligned} 1 &= 5 - 2 * (7 - 1 * 5) \\ &= (-2) * 7 + 3 * 5 \\ &= (-2) * 7 + 3 * (26 - 3 * 7) \\ &= (-11) * 7 + 3 * 26 \end{aligned}$$

Example

Solve: $7x \equiv_{26} 1$

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$26 = 3*7 + 5 \qquad 5 = 26 - 3*7$$
$$7 = 1*5 + 2 \qquad 2 = 7 - 1*5$$
$$5 = 2*2 + 1 \qquad 1 = 5 - 2*2$$

$$
\begin{aligned}
1 &= 5 - 2*(7 - 1*5) \\
&= (-2)*7 + 3*5 \\
&= (-2)*7 + 3*(26 - 3*7) \\
&= (-11)*7 + 3*26
\end{aligned}
$$

Multiplicative inverse of 7 modulo 26

Now $(-11) \bmod 26 = 15$. **So,** $x = 15 + 26k$ **for** $k \in \mathbb{Z}$.

Example of a more general equation

Now solve: $7y \equiv_{26} 3$

We already computed that 15 is the multiplicative inverse of 7 modulo 26. That is, $7 \cdot 15 \equiv_{26} 1$

If y is a solution, then multiplying by 15 we have

$$15 \cdot 7 \cdot y \equiv_{26} 15 \cdot 3$$

Substituting $15 \cdot 7 \equiv_{26} 1$ into this on the left gives

$$\mathrm{y} = 1 \cdot y \equiv_{26} 15 \cdot 3 \equiv_{26} 19$$

This shows that <u>every</u> solution y is congruent to 19.

Example of a more general equation

Now solve: $7y \equiv_{26} 3$

Multiplying both sides of $y \equiv_{26} 19$ by 7 gives

$$7\mathrm{y} \equiv_{26} 7 \cdot 19 \equiv_{26} 3$$

So, <u>any</u> $y \equiv_{26} 19$ is a solution.

Thus, the set of numbers of the form $y = 19 + 26k$, for any k, are <u>exactly</u> solutions of this equation.

Math mod a prime is especially nice

$\gcd(a, m) = 1$ **if** m **is prime and** $0 < a < m$ **so can always solve these equations mod a prime.**

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

mod 7

Multiplicative Inverses and Algebra

Adding to both sides is an equivalence:

$$
\begin{array}{ccc}
 & x \equiv_m y & \\
-c \uparrow & & \downarrow +c \\
 & x + c \equiv_m y + c &
\end{array}
$$

The same is not true of multiplication…

unless we have a multiplicative inverse $cd \equiv_m 1$

$$
\begin{array}{ccc}
 & x \equiv_m y & \\
\times d \uparrow & & \downarrow \times c \\
 & cx \equiv_m cy &
\end{array}
$$