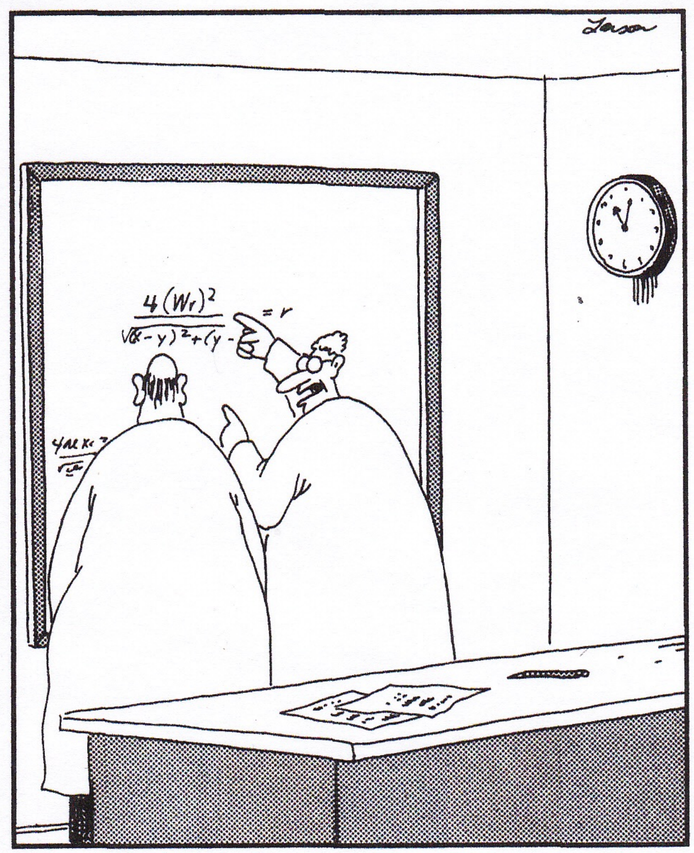
CSE 311: Foundations of Computing

Lecture 9: Proof Strategies & Set Theory



"Yes, yes, I *know* that, Sidney... *every*body knows *that!*... But look: Four wrongs *squared*, minus two wrongs to the fourth power, divided by this formula, *do* make a right."

Last class: Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "The product of two rationals is rational."

OR "If x and y are rational, then xy is rational."

Recall that unquantified variables (not constants) are implicitly for-all quantified.

$\forall x\ \forall y\ ((\text{Rational}(x) \wedge \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Last class: Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (a/b)(c/d) = (ac)/(bd). Since b and d are both non-zero, so is bd. Furthermore, ac and bd are integers. By definition, then, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

Last class: Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Proof: ~~Let x and y be arbitrary rationals.~~

Suppose x and y are rational.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (a/b)(c/d) = (ac)/(bd). Since b and d are both non-zero, so is bd. Furthermore, ac and bd are integers. By definition, then, xy is rational.

~~Since x and y were arbitrary, we have shown that the product of any two rationals is rational.~~ ■

Last class: English Proofs

- High-level language let us work more quickly
  - should not be necessary to spill out every detail
  - examples so far
    - skipping Intro ∧ and Elim ∧ (and hence, Commutativity and Associativity)
    - skipping Double Negation
    - not stating existence claims (immediately apply Elim ∃ to name the object)
    - not stating that the implication has been proven ("Suppose X... Thus, Y." says it already)
  - (list will grow over time)

- English proof is correct if the reader believes they could translate it into a formal proof
  - the reader is the "compiler" for English proofs

Proof Strategies

Proof Strategies: Counterexamples

To prove $\neg\forall x\ P(x)$, prove $\exists\neg P(x)$:

- Equivalent by De Morgan's Law
- All we need to do that is find an x where $P(x)$ is **false**
- This example is called a ***counterexample*** to $\forall x\ P(x)$.

e.g. Prove "Not every prime number is odd"

Proof: 2 is a prime that is not odd — a counterexample to the claim that every prime number is odd. ■

An English proof does not need to cite De Morgan's law.

Proof Strategies: Proof by Contrapositive

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \rightarrow \neg p$, which is equivalent to proving $p \rightarrow q$.

1.1.	$\neg q$	Assumption
...		
1.3.	$\neg p$	
1.	$\neg q \rightarrow \neg p$	Direct Proof
2.	$p \rightarrow q$	Contrapositive: 1

Proof Strategies: Proof by Contrapositive

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \rightarrow \neg p$, which is equivalent to proving $p \rightarrow q$.

We will prove the contrapositive.

Suppose $\neg q$.

...

Thus, $\neg p$.

1.1.	$\neg q$	Assumption
...		
1.3.	$\neg p$	
1.	$\neg q \rightarrow \neg p$	Direct Proof
2.	$p \rightarrow q$	Contrapositive: 1

Proof by Contradiction: One way to prove $\neg p$

If we assume p and derive F (a contradiction), then we have proven $\neg p$.

	1.1. p	Assumption
	...	
	1.3. F	
1.	$p \rightarrow \mathsf{F}$	Direct Proof
2.	$\neg p \vee \mathsf{F}$	Law of Implication: 1
3.	$\neg p$	Identity: 2

Proof Strategies: Proof by Contradiction

If we assume p and derive F (a contradiction), then we have proven $\neg$p.

We will argue by contradiction.

Suppose p.

...

This is a contradiction.

1.1.	p	Assumption
...		
1.3.	F	
1.	$p \rightarrow$ F	Direct Proof
2.	$\neg p \vee$ F	Law of Implication: 1
3.	$\neg p$	Identity: 2

Often, we will infer $\neg$R, where R is a prior fact.
Putting these together, we have R $\wedge$ $\neg$R $\equiv$ F

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

Formally, prove $\neg \exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

Proof: We work by contradiction.

Suppose that x is an integer that is both even and odd. Then, x=2a for some integer a, and x=2b+1 for some integer b. This means 2a=x=2b+1 and hence 2a-2b=1 and so a-b=½. But a-b is an integer while ½ is not, so they cannot be equal. This is a contradiction. ■

Formally, we've shown Integer(½) $\wedge$ $\neg$Integer(½) $\equiv$ F.

Strategies

- Simple proof strategies already do a lot
  - counter examples
  - proof by contrapositive
  - proof by contradiction
- Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)

Applications of Predicate Logic

- Remainder of the course will use predicate logic to prove important properties of interesting objects
  - start with math objects that are widely used in CS
  - eventually more CS-specific objects

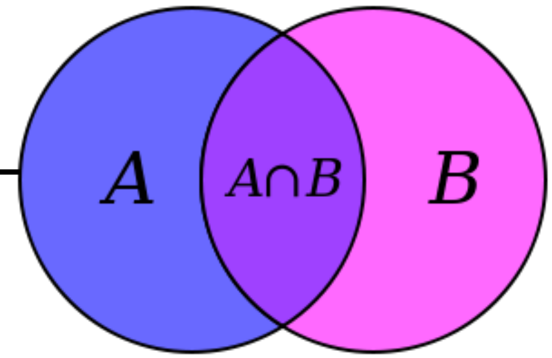
- Encode domain knowledge in predicate definitions
- Then apply predicate logic to infer useful results

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \equiv \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

Set Theory

Set Theory



Sets are collections of objects called **elements.**

Write $a \in B$ to say that a is an element of set B,
and $a \notin B$ to say that it is not.

Some simple examples
A = {1}
B = {1, 3, 2}
C = {□, 1}
D = {{17}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N}$ = {0, 1, 2, …}

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z}$ = {…, -2, -1, 0, 1, 2, …}

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π, $\sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a natural number

∅ = **{}** is the **empty set**; the *only* set with no elements

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\varnothing\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B := \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B := \forall x\,(x \in A \rightarrow x \in B)$$

- Notes:

$$(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})$$

$A \supseteq B$ means $B \subseteq A$

$A \subset B$ means $A \subseteq B$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B := \forall x\, (x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal to each other?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B := \forall\, x\, (x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

QUESTIONS

$\varnothing \subseteq A$?

$A \subseteq B$?

$C \subseteq B$?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B := \forall\, x\, (x \in A \rightarrow x \in B)$$

Note the domain restriction.

We will use a shorthand restriction to a subset

$$\forall x{\in}A\, (P(x)) \;:=\; \forall x\, (x \in A \rightarrow P(x))$$

Building Sets from Predicates

S = the set of all* x for which $P(x)$ is true

$$S = \{x : P(x)\}$$

S = the set of all x in A for which $P(x)$ is true

$$S = \{x \in A : P(x)\}$$

*in the domain of P, usually called the "universe" **U**

Set Operations

$$A \cup B = \{\, x : (x \in A) \vee (x \in B\,)\}$$ **Union**

$$A \cap B = \{\, x : (x \in A) \wedge (x \in B)\}$$ **Intersection**

$$A \setminus B = \{\, x : (x \in A) \wedge (x \notin B)\}$$ **Set Difference**

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] =

{3} =

{1,2} =

More Set Operations

$$A \oplus B = \{ x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

$$\overline{A} = A^C = \{ x : x \notin A \}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

$A \oplus B$ = {3, 4, 6}
$\overline{A}$ = {4,5,6}

Set Complement

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It's remarkable that as recently as 11 years ago, the sum of all human knowledge could be provided in just two books.

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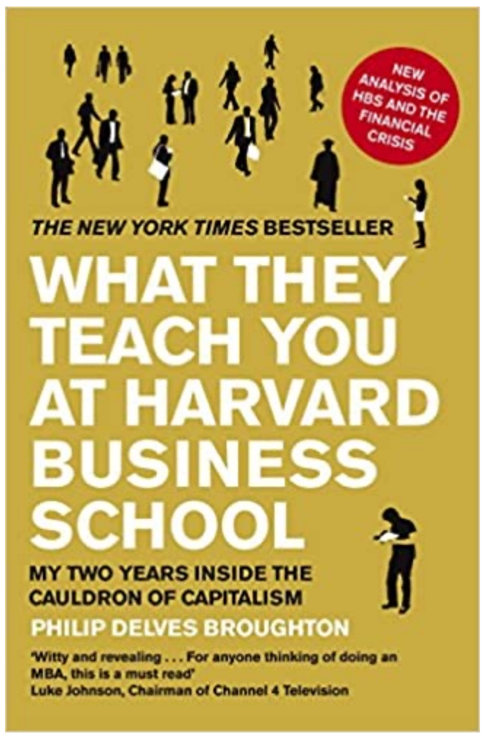

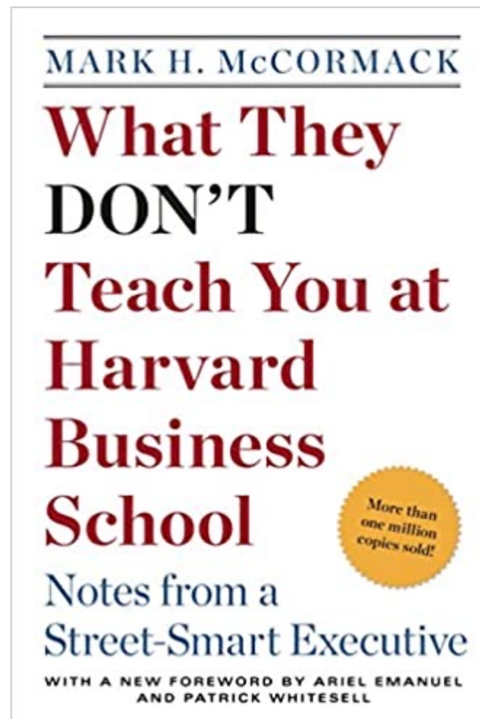

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$