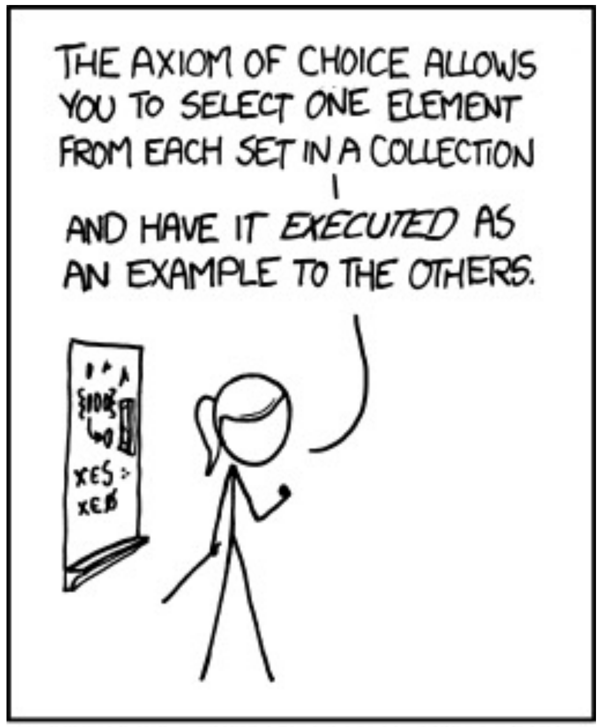
CSE 311: Foundations of Computing

Lecture 9: English Proofs & Proof Strategies



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

Last class: Inference Rules for Quantifiers

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a) \textbf{ for any } a}$$

Intro ∀

$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary}^{*}\text{”}\ldots P(a)}{\therefore\ \forall x\, P(x)}$$

* in the domain of P.

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \textbf{ for some } \textbf{\textit{special}}^{**}\ c}$$

** c is a NEW name.

These rules need some caveats...

There are extra conditions on using these rules:

Intro ∀: "Let a be arbitrary*"...P(a) ∴ ∀x P(x)

Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

* in the domain of P

** c has to be a NEW name.

Over integer domain: ∀x ∃y (y ≥ x) **is True but** ∃y∀x (y ≥ x) **is False**

BAD "PROOF"

1. ∀x ∃y (y ≥ x) — Given
2. **Let a be an arbitrary integer**
3. ∃y (y ≥ **a**) — Elim ∀: **1**
4. **b** ≥ **a** — Elim ∃: **3** (**b**)
5. ∀x (**b** ≥ x) — Intro ∀: **2,4**
6. ∃y∀x (y ≥ x) — Intro ∃ : **5**

These rules need some caveats...

There are extra conditions on using these rules:

Intro $\forall$: "Let a be arbitrary*"...P(a) ∴ $\forall x\, P(x)$

* in the domain of P

Elim $\exists$: $\exists x\, P(x)$ ∴ P(c) for some *special*** c

** c has to be a NEW name.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD "PROOF"

1. $\forall x\, \exists y\, (y \geq x)$ — Given
2. Let **a** be an arbitrary integer
3. $\exists y\, (y \geq a)$ — Elim $\forall$: **1**
4. $b \geq a$ — Elim $\exists$: **3** (**b**)
5. ~~$\forall x\, (b \geq x)$ — Intro $\forall$: 2,4~~
6. $\exists y \forall x\, (y \geq x)$ — Intro $\exists$: **5**

Can't get rid of **a** since another name in the same line, **b**, depends on it!

These rules need some caveats...

There are extra conditions on using these rules:

Intro $\forall$: "Let a be arbitrary*"...P(a) ∴ $\forall x\, P(x)$

* in the domain of P. No other name in P depends on a

Elim $\exists$: $\exists x\, P(x)$ ∴ P(c) for some *special*** c

** c is a NEW name. List all dependencies for c.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD "PROOF"

1. $\forall x\, \exists y\, (y \geq x)$ Given
2. Let a be an arbitrary integer
3. $\exists y\, (y \geq a)$ Elim $\forall$: 1
4. $b \geq a$ Elim $\exists$: 3 (b)
5. ~~$\forall x\, (b \geq x)$ Intro $\forall$: 2,4~~
6. $\exists y \forall x\, (y \geq x)$ Intro $\exists$: 5

Can't get rid of a since another name in the same line, b, depends on it!

Dependencies

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

b depends on a since it appears inside the expression "$\exists y\, (y \geq \mathbf{a})$"

BAD "PROOF"

1.	$\forall x\, \exists y\, (y \geq x)$	Given
2.	**Let a be an arbitrary integer**	
3.	$\exists y\, (y \geq \mathbf{a})$	Elim $\forall$: **1**
4.	$\mathbf{b} \geq \mathbf{a}$	Elim $\exists$: **3** (**b** depends on **a**)
5.	$\forall x\, (\mathbf{b} \geq x)$	Intro $\forall$: **2,4**
6.	$\exists y \forall x\, (y \geq x)$	Intro $\exists$: **5**

Can't Intro $\forall$ **with "Let a be an arbitrary … P(a)"**
because P(a) = "$\mathbf{b} \geq \mathbf{a}$" uses object b, which depends on a!

Dependencies

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

b depends on a since it appears inside the expression "$\exists y\, (y \geq \mathbf{a})$"

BAD "PROOF"

1. $\forall x\, \exists y\, (y \geq x)$ — Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \geq \mathbf{a})$ — Elim $\forall$: **1**
4. $\mathbf{b} \geq \mathbf{a}$ — Elim $\exists$: **3** (**b** depends on **a**)
5. $\forall x\, (\mathbf{b} \geq x)$ — Intro $\forall$: **2,4**
6. $\exists y \forall x\, (y \geq x)$ — Intro $\exists$: **5**

Have instead shown $\forall x\, (\mathbf{b}(x) \geq x)$
where $\mathbf{b}(x)$ **is a number that is possibly different for each** x

Formal Proofs

- In principle, formal proofs are the standard for what it means to be "proven" in mathematics
  - almost all math (and theory CS) done in Predicate Logic

- But they are tedious and impractical
  - e.g., applications of commutativity and associativity
  - Russell & Whitehead's formal proof that 1+1 = 2 is *several hundred pages* long

    we allowed ourselves to cite "Arithmetic", "Algebra", etc.

- Similar situation exists in programming…

Programming

```
a := ADD(i, 1)
b := MOD(a, n)
c := ADD(arr, b)
d := LOAD(c)
e := ADD(arr, i)
STORE(e, d)
```

```
arr[i] = arr[(i+1) % n];
```

Assembly Language **High-level Language**

Programming vs Proofs

`a := ADD(i, 1)`	Given
`b := MOD(a, n)`	Given
`c := ADD(arr, b)`	Elim ∧: 1
`d := LOAD(c)`	Double Negation: 4
`e := ADD(arr, i)`	Elim ∨: 3, 5
`STORE(e, d)`	Modus Ponens: 2, 6
Assembly Language for Programs	**Assembly Language for Proofs**

Proofs

Given

Given

∧ Elim: 1

Double Negation: 4

∨ Elim: 3, 5

MP: 2, 6

what is the "Java" for proofs?

Assembly Language for Proofs

High-level Language for Proofs

Proofs

Given

Given

$\wedge$ Elim: 1

Double Negation: 4

$\vee$ Elim: 3, 5

MP: 2, 6

English?

Assembly Language for Proofs

High-level Language for Proofs

Proofs

Given

Given

∧ Elim: 1

Double Negation: 4

∨ Elim: 3, 5

MP: 2, 6

Math English

Assembly Language for Proofs

High-level Language for Proofs

Proofs

- Formal proofs follow simple well-defined rules and should be easy for a machine to check
  - as assembly language is easy for a machine to execute

- English proofs correspond to those rules but are designed to be easier for humans to read
  - also easy to check with practice

    (almost all actual math and theory CS is done this way)
  - English proof is correct if the <u>reader</u> believes they could translate it into a formal proof

    (the reader is the “compiler” for English proofs)

Last class: Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Prove: "The square of every even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	∃y (**a** = 2y)	Definition of Even
2.3	**a** = 2**b**	Elim ∃: **b** special depends on **a**
2.4	$a^2 = 4b^2 = 2(2b^2)$	Algebra
2.5	∃y (a^2 = 2y)	Intro ∃ rule
2.6	Even(a^2)	Definition of Even

2. Even(**a**)→Even(a^2) — Direct Proof
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

English Proof: Even and Odd

$Even(x) \equiv \exists y\ (x=2y)$
$Odd(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Let **a** be an arbitrary integer.	1. Let **a** be an arbitrary integer
Suppose **a** is even.	2.1 Even(**a**) — Assumption
Then, by definition, $a = 2b$ for some integer **b** (dep on **a**).	2.2 $\exists y\ (a = 2y)$ — Definition 2.3 $a = 2b$ — **b** special depends on **a**
Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$.	2.4 $a^2 = 4b^2 = 2(2b^2)$ — Algebra
So a^2 is, by definition, even.	2.5 $\exists y\ (a^2 = 2y)$ 2.6 $Even(a^2)$ — Definition
Since **a** was arbitrary, we have shown that the square of every even number is even.	2. $Even(a) \rightarrow Even(a^2)$ 3. $\forall x\ (Even(x) \rightarrow Even(x^2))$

English Proof: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let **a** be an arbitrary integer.

Suppose **a** is even. Then, by definition, **a** = 2**b** for some integer b (depending on **a**). Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$. So a^2 is, by definition, is even.

Since **a** was arbitrary, we have shown that the square of every even number is even. ■

English Proof: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let **a** be an arbitrary **even** integer.

Then, by definition, **a** = 2**b** for some integer b (dep on **a**). Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$. So a^2 is, by definition, is even.

Since **a** was arbitrary, we have shown that the square of every even number is even. ■

$$\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$$

Even and Odd

Predicate Definitions

Even(x) ≡ $\exists y\ (x = 2y)$

Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse

Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y))$

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y))$

Let x and y be arbitrary integers.

1. **Let x be an arbitrary integer**
2. **Let y be an arbitrary integer**

Since x and y were arbitrary, the sum of any odd integers is even.

3. $(\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})) \rightarrow \text{Even}(\mathbf{x+y})$
4. $\forall x\ \forall y\ ((\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})) \rightarrow \text{Even}(x+y))$ Intro $\forall$

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove $\forall x\ \forall y\ ((\text{Odd}(x) \land \text{Odd}(y)) \rightarrow \text{Even}(x+y))$

Let x and y be arbitrary integers.	1. **Let x be an arbitrary integer** 2. **Let y be an arbitrary integer**
Suppose that both are odd.	**3.1** Odd(**x**) ∧ Odd(**y**) Assumption
so x+y is even.	**3.9** Even(**x+y**)
Since x and y were arbitrary, the sum of any odd integers is even.	**3.** (Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**) DPR **4.** ∀x ∀y ((Odd(**x**) ∧ Odd(**y**)) → Even(x+y)) Intro ∀

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Formally, prove ∀x ∀y ((Odd(x) ∧ Odd(y))→Even(x+y))

Let x and y be arbitrary integers.

1. Let **x** be an arbitrary integer
2. Let **y** be an arbitrary integer

Suppose that both are odd.

3.1	Odd(**x**) ∧ Odd(**y**)	Assumption
3.2	Odd(**x**)	Elim ∧: 2.1
3.3	Odd(**y**)	Elim ∧: 2.1

so x+y is even.

3.9	Even(**x+y**)	

Since x and y were arbitrary, the sum of any odd integers is even.

3.	(Odd(**x**) ∧ Odd(**y**)) → Even(**x+y**)	DPR
4.	∀x ∀y ((Odd(**x**) ∧ Odd(**y**)) → Even(x+y))	Intro ∀

English Proof: Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Prove "The sum of two odd numbers is even."

Let x and y be arbitrary integers.	**1. Let x be an arbitrary integer** **2. Let y be an arbitrary integer**
Suppose that both are odd.	3.1 Odd(x) ∧ Odd(y) — Assumption 3.2 Odd(x) — Elim ∧: 2.1 3.3 Odd(y) — Elim ∧: 2.1
Then, x = 2a+1 for some integer a (depending on x) and y = 2b+1 for some integer b (depending on y).	3.4 ∃z (x = 2z+1) — Def of Odd: 2.2 3.5 x = 2a+1 — Elim ∃: 2.4 (a dep x) 3.6 ∃z (y = 2z+1) — Def of Odd: 2.3 3.7 y = 2b+1 — Elim ∃: 2.5 (b dep y)
so x+y is, by definition, even.	3.9 ∃z (x+y = 2z) — Intro ∃: 2.4 3.10 Even(x+y) — Def of Even
Since x and y were arbitrary, the sum of any odd integers is even.	3. (Odd(x) ∧ Odd(y)) → Even(x+y) — DPR 4. ∀x ∀y ((Odd(x) ∧ Odd(y)) → Even(x+y)) — Intro ∀

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The sum of two odd numbers is even."

Let x and y be arbitrary integers.	1.	Let **x** be an arbitrary integer	
	2.	Let **y** be an arbitrary integer	
Suppose that both are odd.	3.1	$\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})$	Assumption
	3.2	$\text{Odd}(\mathbf{x})$	Elim ∧: 2.1
	3.3	$\text{Odd}(\mathbf{y})$	Elim ∧: 2.1
Then, x = 2a+1 for some integer a (depending on x) and	3.4	$\exists \mathbf{z}\ (\mathbf{x} = 2\mathbf{z}+1)$	Def of Odd: 2.2
	3.5	$\mathbf{x} = 2\mathbf{a}+1$	Elim ∃: 2.4 (**a** dep **x**)
y = 2b+1 for some integer b (depending on y).	3.6	$\exists \mathbf{z}\ (\mathbf{y} = 2\mathbf{z}+1)$	Def of Odd: 2.3
	3.7	$\mathbf{y} = 2\mathbf{b}+1$	Elim ∃: 2.5 (**b** dep **y**)
Their sum is x+y = ... = 2(a+b+1)	3.8	$\mathbf{x+y} = 2(\mathbf{a}+\mathbf{b}+\mathbf{1})$	Algebra
so x+y is, by definition, even.	3.9	$\exists z\ (\mathbf{x+y} = 2z)$	Intro ∃: 2.4
	3.10	$\text{Even}(\mathbf{x+y})$	Def of Even
Since x and y were arbitrary, the sum of any odd integers is even.	3.	$(\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})) \rightarrow \text{Even}(\mathbf{x+y})$	DPR
	4.	$\forall x\ \forall y\ ((\text{Odd}(\mathbf{x}) \land \text{Odd}(\mathbf{y})) \rightarrow \text{Even}(\mathbf{x+y}))$	Intro ∀

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Proof: Let x and y be arbitrary integers.

Suppose that both are odd. Then, x = 2a+1 for some integer a (depending on x) and y = 2b+1 for some integer b (depending on x). Their sum is x+y = (2a+1) + (2b+1) = 2a+2b+2 = 2(a+b+1), so x+y is, by definition, even.

Since x and y were arbitrary, the sum of any two odd integers is even. ■

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The sum of two odd numbers is even."

Proof: Let x and y be arbitrary **odd** integers.

Then, x = 2a+1 for some integer a (depending on x) and y = 2b+1 for some integer b (depending on x). Their sum is x+y = (2a+1) + (2b+1) = 2a+2b+2 = 2(a+b+1), so x+y is, by definition, even.

Since x and y were arbitrary, the sum of any two odd integers is even. ■

$\forall x\ \forall y\ ((\mathrm{Odd}(x) \land \mathrm{Odd}(y)) \rightarrow \mathrm{Even}(x+y))$

Rational Numbers

Domain of Discourse
Real Numbers

- A real number x is *rational* iff there exist integers a and b with $b \neq 0$ such that $x = a/b$.

$$\text{Rational}(x) := \exists a\, \exists b\, (((\text{Integer}(a) \land \text{Integer}(b)) \land (x = a/b)) \land b \neq 0)$$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Formally, prove $\forall x\ \forall y\ ((\text{Rational}(x) \land \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "The product of two rationals is rational."

Proof: Let x and y be arbitrary rationals.

Then, $x = a/b$ for some integers a, b, where $b \neq 0$, and $y = c/d$ for some integers c,d, where $d \neq 0$.

Multiplying, we get that $xy = (a/b)(c/d) = (ac)/(bd)$. Since b and d are both non-zero, so is bd. Furthermore, ac and bd are integers. By definition, then, xy is rational.

Since x and y were arbitrary, we have shown that the product of any two rationals is rational. ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "The product of two rationals is rational."

OR "If x and y are rational, then xy is rational."

Recall that unquantified variables (not constants) are implicitly for-all quantified.

$\forall x\ \forall y\ ((\text{Rational}(x) \wedge \text{Rational}(y)) \rightarrow \text{Rational}(xy))$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Proof: ~~Let x and y be arbitrary rationals.~~
Suppose x and y are rational.

Then, x = a/b for some integers a, b, where b≠0, and
y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (a/b)(c/d) = (ac)/(bd).
Since b and d are both non-zero, so is bd. Furthermore,
ac and bd are integers. By definition, then, xy is rational.

~~Since x and y were arbitrary, we have shown that the product of any two rationals is rational.~~ ■

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.

1.1 $\text{Rational}(x) \land \text{Rational}(y)$ **Assumption**

Then, x = a/b for some integers a, b, where b≠0 and y = c/d for some integers c,d, where d≠0.

...

1.4 $\exists p\, \exists q\, ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$
Def Rational: 1.2

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$
Elim ∃: 1.4

1.6 $\exists p\, \exists q\, ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$
Def Rational: 1.3

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$
Elim ∃: 1.4

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.

1.1 $\text{Rational}(x) \wedge \text{Rational}(y)$ **Assumption**

??

Then, x = a/b for some integers a, b, where b≠0 and y = c/d for some integers c,d, where d≠0.

...

1.4 $\exists p\ \exists q\ ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$
Def Rational: 1.2

1.5 $(x = a/b) \wedge \text{Integer}(a) \wedge \text{Integer}(b) \wedge (b \neq 0)$
Elim ∃: 1.4

1.6 $\exists p\ \exists q\ ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$
Def Rational: 1.3

1.7 $(y = c/d) \wedge \text{Integer}(c) \wedge \text{Integer}(d) \wedge (d \neq 0)$
Elim ∃: 1.4

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.

Then, x = a/b for some integers a, b, where b≠0 and y = c/d for some integers c,d, where d≠0.

...

1.1 $\text{Rational}(x) \land \text{Rational}(y)$ **Assumption**

1.2 $\text{Rational}(x)$ **Elim ∧: 1.1**

1.3 $\text{Rational}(y)$ **Elim ∧: 1.1**

1.4 $\exists p\, \exists q\, ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$ **Def Rational: 1.2**

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$ **Elim ∃: 1.4**

1.6 $\exists p\, \exists q\, ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$ **Def Rational: 1.3**

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$ **Elim ∃: 1.4**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$

...

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$

Multiplying, we get xy = (ac)/(bd).

1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$ **Algebra**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

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??

Multiplying, we get xy = (ac)/(bd).

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Rationality

Domain of Discourse
Real Numbers

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…
1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$
…
1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$
1.8 $x = a/b$ **Elim ∧: 1.5**
1.9 $y = c/d$ **Elim ∧: 1.7**
1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$ **Algebra**

Multiplying, we get xy = (ac)/(bd).

Rationality

Domain of Discourse
Real Numbers

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Rational(x) := $\exists a \, \exists b \, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

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...

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$

...

	1.11	$b \neq 0$	**Elim ∧: 1.5***
	1.12	$d \neq 0$	**Elim ∧: 1.7**
Since b and d are non-zero, so is bd.	**1.13**	$bd \neq 0$	**Prop of Integer Mult**

*** Oops, I skipped steps here...**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

…

1.5 $(x = a/b) \wedge (\text{Integer}(a) \wedge (\text{Integer}(b) \wedge (b \neq 0)))$

…

1.7 $(y = c/d) \wedge (\text{Integer}(c) \wedge (\text{Integer}(d) \wedge (d \neq 0)))$

…

1.11 $\text{Integer}(a) \wedge \left(\text{Integer}(b) \wedge (b \neq 0)\right)$ **Elim ∧: 1.5**

1.12 $\text{Integer}(b) \wedge (b \neq 0)$ **Elim ∧: 1.11**

1.13 $b \neq 0$ **Elim ∧: 1.12**

We left out the parentheses…

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

...

1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$

...

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$

...

1.13 $b \neq 0$ — **Elim ∧: 1.5**

...

1.16 $d \neq 0$ — **Elim ∧: 1.7**

Since b and d are non-zero, so is bd.

1.17 $bd \neq 0$ — **Prop of Integer Mult**

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

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1.5 $(x = a/b) \land \text{Integer}(a) \land \text{Integer}(b) \land (b \neq 0)$

...

1.7 $(y = c/d) \land \text{Integer}(c) \land \text{Integer}(d) \land (d \neq 0)$

...

1.19 $\text{Integer}(a)$ — **Elim ∧: 1.5***

...

1.22 $\text{Integer}(b)$ — **Elim ∧: 1.5***

...

1.24 $\text{Integer}(c)$ — **Elim ∧: 1.7***

...

1.27 $\text{Integer}(d)$ — **Elim ∧: 1.7***

1.28 $\text{Integer}(ac)$ — **Prop of Integer Mult**

1.29 $\text{Integer}(bd)$ — **Prop of Integer Mult**

Furthermore, ac and bd are integers.

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
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Prove: "If x and y are rational, then xy is rational."

...

1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$

...

1.17 $bd \neq 0$ **Prop of Integer Mult**

...

1.28 $\text{Integer}(ac)$ **Prop of Integer Mult**

1.29 $\text{Integer}(bd)$ **Prop of Integer Mult**

1.30 $\text{Integer}(bd) \wedge (bd \neq 0)$ **Intro ∧: 1.29, 1.17**

1.31 $\text{Integer}(ac) \wedge \text{Integer}(bd) \wedge (bd \neq 0)$
Intro ∧: 1.28, 1.30

1.32 $(xy = (a/b)/(c/d)) \wedge \text{Integer}(ac) \wedge \text{Integer}(bd) \wedge (bd \neq 0)$ **Intro ∧: 1.10, 1.31**

1.33 $\exists p\ \exists q\ ((xy = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$
Intro ∃: 1.32

1.34 $\text{Rational}(xy)$ **Def of Rational: 1.32**

By definition, then, xy is rational.

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose x and y are rational.	**1.1** $\text{Rational}(x) \land \text{Rational}(y)$	**Assumption**
	...	
	1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$	
	...	
	1.17 $bd \neq 0$	**Prop of Integer Mult**
	...	
Furthermore, ac and bd are integers.	**1.28** $\text{Integer}(ac)$	**Prop of Integer Mult**
	1.29 $\text{Integer}(bd)$	**Prop of Integer Mult**
	...	
By definition, then, xy is rational.	**1.34** $\text{Rational}(xy)$	**Def of Rational: 1.32**

And finally…

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\ \exists b\ (\text{Integer}(a) \wedge \text{Integer}(b) \wedge (x = a/b) \wedge (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Suppose that x and y are rational.	**1.1** $\text{Rational}(x) \wedge \text{Rational}(y)$ **Assumption**
	...
	1.10 $xy = (a/b)(c/d) = (ac/bd) = (ac)/(bd)$
	...
	1.17 $bd \neq 0$ **Prop of Integer Mult**
	...
	1.28 $\text{Integer}(ac)$ **Prop of Integer Mult**
Furthermore, ac and bd are integers.	**1.29** $\text{Integer}(bd)$ **Prop of Integer Mult**
	...
By definition, then, xy is rational.	**1.34** $\text{Rational}(xy)$ **Def of Rational: 1.32**

1. $\text{Rational}(x) \wedge \text{Rational}(y) \rightarrow \text{Rational}(xy)$

Direct Proof

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) := $\exists a\, \exists b\, (\text{Integer}(a) \land \text{Integer}(b) \land (x = a/b) \land (b \neq 0))$

Prove: "If x and y are rational, then xy is rational."

Proof: Suppose x and y are rational.

Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (ac)/(bd). Since b and d are both non-zero, so is bd. Furthermore, ac and bd are integers. By definition, then, xy is rational. ■

vs more than 35 lines of formal proof

English Proofs

- High-level language let us work more quickly
  - should not be necessary to spill out every detail
  - reader checks that the writer is not skipping too much
  - examples so far
    - skipping Intro ∧ and Elim ∧
    - not stating existence claims (immediately apply Elim ∃ to name the object)
    - not stating that the implication has been proven ("Suppose X... Thus, Y." says it already)
  - (list will grow over time)

- English proof is correct if the reader believes they could translate it into a formal proof
  - the reader is the "compiler" for English proofs