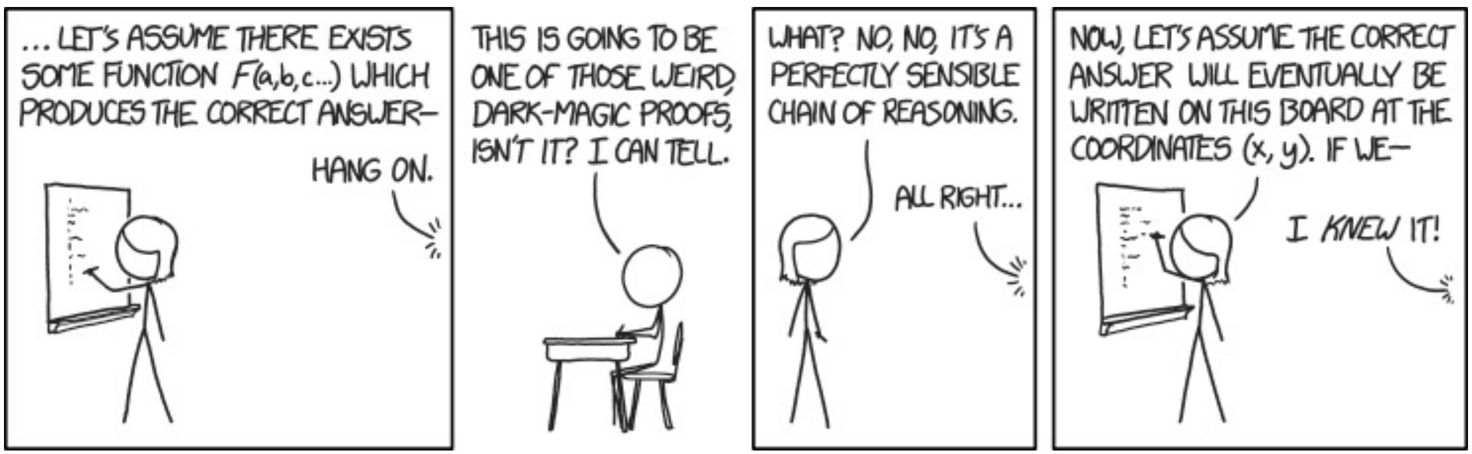
CSE 311: Foundations of Computing

Lecture 8: Predicate Logic Proofs



Last class: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A, B}$$

Intro ∧

$$\frac{A \,;\, B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B \,;\, \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens

$$\frac{A \,;\, A \rightarrow B}{\therefore B}$$

Direct Proof Rule

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

Last Class: To Prove An Implication: $A \rightarrow B$

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is **a proof** that "**Given** A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \rightarrow B$ is true

To Prove An Implication: $A \to B$

$$\frac{A \Rightarrow B}{\therefore A \to B}$$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is a **proof** that "**Given** A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \to B$ is true

Example: Prove $p \to (p \vee q)$.

proof subroutine

Indent proof subroutine ⇒

1.1.	p	Assumption
1.2.	$p \vee q$	Intro $\vee$: 1
1.	$p \to (p \vee q)$	Direct Proof Rule

Proofs using the direct proof rule

Show that $p \rightarrow r$ follows from q and $(p \wedge q) \rightarrow r$

1. q Given
2. $(p \wedge q) \rightarrow r$ Given

This is a proof of $p \rightarrow r$

3.1.	p	Assumption
3.2.	$p \wedge q$	Intro $\wedge$: 1, 3.1
3.3.	r	MP: 2, 3.2

If we know p is true… Then, we've shown r is true

3. $p \rightarrow r$ Direct Proof Rule

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens...

Example

Prove: $(p \land q) \rightarrow (p \lor q)$

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1. $p \wedge q$ — Assumption

1.? $p \vee q$

1. $(p \wedge q) \rightarrow (p \vee q)$ — Direct Proof Rule

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

1.1. $p \wedge q$ Assumption

1.2. p Elim $\wedge$: 1.1

1.? $p \vee q$

1. $(p \wedge q) \rightarrow (p \vee q)$ Direct Proof Rule

Example

Prove: $(p \land q) \rightarrow (p \lor q)$

1.1.	$p \land q$	Assumption
1.2.	p	Elim $\land$: 1.1
1.3.	$p \lor q$	Intro $\lor$: 1.2
1.	$(p \land q) \rightarrow (p \lor q)$	Direct Proof Rule

One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1. $(p \to q) \land (q \to r)$ Assumption

1.? $p \to r$

1. $\big((p \to q) \land (q \to r)\big) \to (p \to r)$ Direct Proof Rule

Example

Prove: $((p \to q) \wedge (q \to r)) \to (p \to r)$

1.1.	$(p \to q) \wedge (q \to r)$	Assumption
1.2.	$p \to q$	$\wedge$ Elim: 1.1
1.3.	$q \to r$	$\wedge$ Elim: 1.1
1.?	$p \to r$	
1.	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Direct Proof Rule

Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1.	$(p \to q) \land (q \to r)$	Assumption
1.2.	$p \to q$	$\land$ Elim: 1.1
1.3.	$q \to r$	$\land$ Elim: 1.1
1.4.1.	p	Assumption
1.4.?	r	
1.4.	$p \to r$	Direct Proof Rule
1.	$((p \to q) \land (q \to r)) \to (p \to r)$	Direct Proof Rule

Example

Prove: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

1.1. $(p \rightarrow q) \wedge (q \rightarrow r)$ Assumption

1.2. $p \rightarrow q$ $\wedge$ Elim: 1.1

1.3. $q \rightarrow r$ $\wedge$ Elim: 1.1

1.4.1. p Assumption

1.4.2. q MP: 1.2, 1.4.1

1.4.3. r MP: 1.3, 1.4.2

1.4. $p \rightarrow r$ Direct Proof Rule

1. $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$ Direct Proof Rule

Inference Rules for Quantifiers: First look

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a)}$$ **(for any** a)

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c)}$$ **for some *special*** ** c

Intro ∀

** By special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) **for any** a

Prove $(\forall x\, P(x)) \rightarrow (\exists x\, P(x))$

5. $(\forall x\, P(x)) \rightarrow (\exists x\, P(x))$?

The main connective is implication so Direct Proof Rule seems good

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have
so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$?

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: $\dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$

Elim ∀: $\dfrac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have
so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$ Intro ∃: ?

That requires P(c)
for some c.

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: $\dfrac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$

Elim ∀: $\dfrac{\forall x\, P(x)}{\therefore \quad P(a) \textbf{ for any } a}$

Prove $\forall x\ P(x) \rightarrow \exists x\ P(x)$

1.1. $\forall x\ P(x)$ **Assumption**

1.2. **Let a be an object.**

1.5. $\exists x\ P(x)$ **Intro ∃: ?**

1. $\forall x\ P(x) \rightarrow \exists x\ P(x)$ **Direct Proof Rule**

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

1.2. Let a be an object.

1.4. $P(a)$?

1.5. $\exists x\, P(x)$ Intro ∃: 1.4

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃ $\dfrac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$

1.1.	$\forall x\, P(x)$	Assumption
1.2.	Let a be an object.	
1.4.	$P(a)$	Elim ∀: 1.1
1.5.	$\exists x\, P(x)$	Intro ∃: 1.4

Prove $\forall x\, P(x) \to \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

1.2. Let a be an object.

1.4. $P(a)$ Elim ∀: 1.1

1.5. $\exists x\, P(x)$ Intro ∃: 1.4

1. $\forall x\, P(x) \to \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove ∀x P(x) → ∃x P(x)

1.1.	$\forall x\, P(x)$	Assumption
1.2.	Let a be an object.	
1.3.	$P(a)$	Elim ∀: 1.1
1.4.	$\exists x\, P(x)$	Intro ∃: 1.3

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

Working forwards as well as backwards:
In applying "Intro ∃" rule we didn't know what expression we might be able to prove P(c) for, so we worked forwards to figure out what might work.

Predicate Logic Proofs

- Can use
  - Predicate logic inference rules

    whole formulas only
  - Predicate logic equivalences (De Morgan's)

    even on subformulas
  - Propositional logic inference rules

    whole formulas only
  - Propositional logic equivalences

    even on subformulas

Predicate Logic Proofs with more content

- In propositional logic we could just write down other propositional logic statements as "givens"

- Here, we also want to be able to use domain knowledge so proofs are about something specific

- Example:

Domain of Discourse
Integers

- Given the basic properties of arithmetic on integers, define:

Predicate Definitions
Even(x) := $\exists$y (x = 2·y)
Odd(x) := $\exists$y (x = 2·y + 1)

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := $\exists y\ (x = 2 \cdot y)$
Odd(x) := $\exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := $\exists y\ (x = 2 \cdot y)$
Odd(x) := $\exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

1.	$2 = 2 \cdot 1$	**Algebra**
2.	$\exists y\ (2 = 2 \cdot y)$	**Intro** ∃**: 1**
3.	Even(**2**)	**Definition of** Even**: 2**
4.	$\exists x\ \text{Even}(x)$	**Intro** ∃**: 3**

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := $\exists$y (x = 2·y)
Odd(x) := $\exists$y (x = 2·y + 1)
Prime(x) := "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove $\exists x\,(Even(x) \wedge Prime(x))$

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) := ∃y (x = 2·y) Odd(x) := ∃y (x = 2·y + 1) Prime(x) := "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove ∃x (Even(x) ∧ Prime(x))

1.	**2 = 2·1**	**Algebra**
2.	∃y (**2** = **2**·y)	**Intro ∃: 1**
3.	Even(**2**)	**Def of** Even**: 3**
4.	Prime(**2**)*	**Property of integers**
5.	Even(**2**) ∧ Prime(**2**)	**Intro ∧: 2, 4**
6.	∃x (Even(x) ∧ Prime(x))	**Intro ∃: 5**

*** Later we will further break down** "Prime" **using quantifiers to prove statements like this**

Inference Rules for Quantifiers: First look

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \quad \textbf{(for any } a\textbf{)}}$$

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \textbf{ for some } \textbf{\textit{special}}^{**}\ c}$$

Intro ∀

$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary}^{*}\text{”} \ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

** By special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!

* in the domain of P

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
"**Let** a **be arbitrary***"...P(a) ∴ ∀x P(x)	∃x P(x) ∴ P(c) **for some *special*** ** c

Prove: "The square of any even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

3. ∀x (Even(x)→Even(x^2)) ?

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let **a** be an arbitrary integer

2. Even(**a**)→Even($\mathbf{a}^2$) ?
3. ∀x (Even(x)→Even(x^2)) Intro ∀: 1,2

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

1. Let a be an arbitrary integer
2.1 Even(a) Assumption
2.6 Even(a^2)
2. Even(a)→Even(a^2) Direct proof rule
3. ∀x (Even(x)→Even(x^2)) Intro ∀: 1,2

1. Let a be an arbitrary integer
2.1 Even(a) Assumption
2.6 Even(a^2)
2. Even(a)→Even(a^2) Direct proof rule
3. ∀x (Even(x)→Even(x^2)) Intro ∀: 1,2

Prove: "The square of any even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let **a** be an arbitrary integer
   2.1 Even(**a**) Assumption

   2.6 Even($\mathbf{a}^2$) ?
2. Even(**a**)→Even($\mathbf{a}^2$) Direct proof rule
3. ∀x (Even(x)→Even(x^2)) Intro ∀: 1,2

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀: "Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

   2.1 Even(**a**) — Assumption
   2.2 $\exists y\,(\mathbf{a} = 2y)$ — Definition of Even

   2.5 $\exists y\,(\mathbf{a}^2 = 2y)$ — ?
   2.6 Even($\mathbf{a}^2$) — Definition of Even

2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof rule
3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$ — Intro ∀: 1,2

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀		Elim ∃	
	"Let a be arbitrary*"...P(a)		∃x P(x)
	∴ ∀x P(x)		∴ P(c) for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 ∃y (**a** = 2y) — Definition of Even
   - 2.5 ∃y (**a**^{2} = 2y) — Intro ∃ rule: ?
   - 2.6 Even(**a**^{2}) — Definition of Even
2. Even(**a**)→Even(**a**^{2}) — Direct proof rule
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

Need **a**^{2} = 2**c** for some **c**

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 ∃y (**a** = 2y) — Definition of Even
   - 2.3 **a** = 2**b** — Elim ∃: **b**
   - 2.5 ∃y ($\mathbf{a}^2$ = 2y) — Intro ∃ rule: ?
   - 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof rule
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

Need $\mathbf{a}^2 = 2\mathbf{c}$ for some **c**

Even and Odd

Even(x) := ∃y (x=2y)
Odd(x) := ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of any even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	$\exists y\,(\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim ∃: **b**
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\,(\mathbf{a}^2 = 2y)$	Intro ∃ rule
2.6	Even($\mathbf{a}^2$)	Definition of Even

Used $\mathbf{a}^2 = 2c$ for $c=2\mathbf{b}^2$

2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof rule
3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$ — Intro ∀: 1,2

These rules need more caveats…

There are extra conditions on using these rules:

Intro ∀: "**Let** a **be arbitrary***"…P(a) ∴ $\forall x\, P(x)$

* in the domain of P

Elim ∃: $\exists x\, P(x)$ ∴ P(c) **for some *special***** c

** c has to be a NEW name.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD "PROOF"

1. $\forall x\, \exists y\, (y \geq x)$ Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \geq \mathbf{a})$ Elim ∀: **1**
4. $\mathbf{b} \geq \mathbf{a}$ Elim ∃: **b**
5. $\forall x\, (\mathbf{b} \geq x)$ Intro ∀: **2,4**
6. $\exists y \forall x\, (y \geq x)$ Intro ∃ : **5**

These rules need more caveats…

There are extra conditions on using these rules:

Intro ∀: “**Let** a **be arbitrary***”…P(a) ∴ $\forall x\, P(x)$

* in the domain of P

Elim ∃: $\exists x\, P(x)$ ∴ P(c) **for some *special*** ** c

** c has to be a NEW name.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD “PROOF”

1. $\forall x\, \exists y\, (y \geq x)$ — Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \geq \mathbf{a})$ — Elim ∀: **1**
4. $\mathbf{b} \geq \mathbf{a}$ — Elim ∃: **b**
5. $\forall x\, (\mathbf{b} \geq x)$ — Intro ∀: **2,4**
6. $\exists y \forall x\, (y \geq x)$ — Intro ∃ : **5**

Can’t get rid of a since another name in the same line, b, depends on it!

These rules need more caveats…

There are extra conditions on using these rules:

Intro ∀: "Let a be arbitrary*"…P(a) ∴ ∀x P(x)

* in the domain of P. No other name in P depends on a

Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

** c is a NEW name. List all dependencies for c.

Over integer domain: ∀x ∃y (y ≥ x) is True but ∃y∀x (y ≥ x) is False

BAD "PROOF"

1. ∀x ∃y (y ≥ x) Given
2. Let **a** be an arbitrary integer
3. ∃y (y ≥ **a**) Elim ∀: 1
4. **b** ≥ **a** Elim ∃: **b** special depends on **a**
5. ~~∀x (**b** ≥ x) Intro ∀: 2,4~~
6. ∃y∀x (y ≥ x) Intro ∃ : 5

Can't get rid of **a** since another name in the same line, **b**, depends on it!

Inference Rules for Quantifiers: Full version

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a) \textbf{ for any } a}$$

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \textbf{ for some } \textbf{\textit{special}}^{**}\ c}$$

** c is a NEW name.
List all dependencies for c.

Intro ∀

$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary}^{*}\text{”}\ldots P(a)}{\therefore\ \forall x\, P(x)}$$

* in the domain of P. No other name in P depends on a

English Proofs

- We often write proofs in English rather than as fully formal proofs
  - They are more natural to read

- English proofs follow the structure of the corresponding formal proofs
  - Formal proof methods help to understand how proofs really work in English…

    … and give clues for how to produce them.