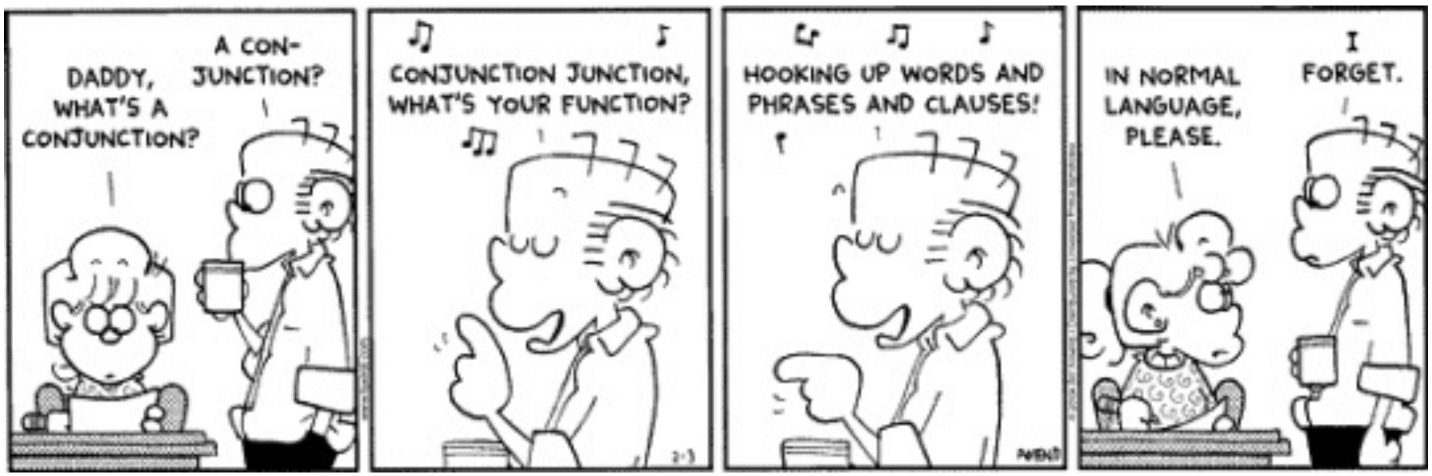
CSE 311: Foundations of Computing

Lecture 2: More Logic, Equivalence & Digital Circuits



Homework

- Homework released Saturdays, due on Fridays
  - usually includes some material from Monday
  - usually 6 problems (+1 feedback +1 extra credit)

- Consider doing 1 problem per day
  - if the material is clear, problems hopefully take 20-30 minutes
  - if the material is unclear, it will take longer to review, ask Qs, etc.
  - much better to find out earlier in the week what is unclear

- **HW1** is out now (a day early)

- Gradescope invites should go out on Monday

Last class: Atomic Propositions

Simplest units (words) in this logical language

Propositional Variables: $p, q, r, s, \ldots$

Truth Values:

- **T** for **true**
- **F** for **false**

Last class: Some Connectives & Truth Tables

Negation (not)

p	$\neg p$
T	F
F	T

Conjunction (and)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (or)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Last class: Implication

"If it's raining, then I have my umbrella"

p	r	$p \to r$
T	T	T
T	F	F
F	T	T
F	F	T

In English, we can also write

"I have my umbrella if it's raining."

$p \rightarrow r$

(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

$p \rightarrow r$

(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

These sentences are implications in opposite directions:

(1) "Pokémon masters have all 151 Pokémon"

(2) "People who have 151 Pokémon are Pokémon masters"

So, the implications are:

(1) If I am a Pokémon master, *then* I have collected all 151 Pokémon.

(2) If I have collected all 151 Pokémon, *then* I am a Pokémon master.

$p \rightarrow r$

Implication:

- *p* implies *r*
- whenever *p* is true *r* must be true
- if *p* then *r*
- *r* if *p*
- *p* only if *r*
- *p* is sufficient for *r*
- *r is necessary for p*

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional: $p \leftrightarrow r$

- *p* if and only if *r* (*p* iff *r*)
- *p* implies *r* and *r* implies *p*
- *p* is necessary and sufficient for *r*

p	r	$p \leftrightarrow r$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional: $p \leftrightarrow r$

- *p* if and only if *r*   (*p* iff *r*)
- *p* implies *r* and *r* implies *p*
- *p* is necessary and sufficient for *r*

p	r	$p \leftrightarrow r$	$p \rightarrow r$	$r \rightarrow p$	$(p \rightarrow r) \wedge (r \rightarrow p)$
T	T	T	T	T	
T	F	F	F	T	
F	T	F	T	F	
F	F	T	T	T	

Biconditional: $p \leftrightarrow r$

- *p* if and only if *r* (*p* iff *r*)
- *p* implies *r* and *r* implies *p*
- *p* is necessary and sufficient for *r*

p	r	$p \leftrightarrow r$	$p \rightarrow r$	$r \rightarrow p$	$(p \rightarrow r) \wedge (r \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Back to Garfield…

q	**"Garfield has black stripes"**
r	**"Garfield is an orange cat"**
s	**"Garfield likes lasagna"**

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

↓

(q if (r and s)) and (r or (not s))

↓

(q "if" $(r \land s)) \land$ **(r** $\lor \neg$**s)**

Back to Garfield...

q	"Garfield has black stripes"
r	"Garfield is an orange cat"
s	"Garfield likes lasagna"

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

(q if (r and s)) and (r or (not s))

$(\mathbf{q}\ \text{“}\mathbf{if}\text{”}\ (r \land s)) \land (\mathbf{r} \lor \neg \mathbf{s})$

$((r \land s) \longrightarrow \mathbf{q}) \land (\mathbf{r} \lor \neg \mathbf{s})$

Analyzing the Garfield Sentence with a Truth Table

q	r	r	$((q \wedge r) \rightarrow p) \wedge (q \vee \neg r)$
F	F	F	
F	F	T	
F	T	F	
F	T	T	
T	F	F	
T	F	T	
T	T	F	
T	T	T	

Analyzing the Garfield Sentence with a Truth Table

q	r	s	$r \vee \neg s$	$(r \wedge s) \to q$	$((r \wedge s) \to q) \wedge (r \vee \neg s)$
F	F	F			
F	F	T			
F	T	F			
F	T	T			
T	F	F			
T	F	T			
T	T	F			
T	T	T			

Analyzing the Garfield Sentence with a Truth Table

q	r	s	$\neg s$	$r \vee \neg s$	$r \wedge s$	$(r \wedge s) \to q$	$((r \wedge s) \to q) \wedge (r \vee \neg s)$
F	F	F					
F	F	T					
F	T	F					
F	T	T					
T	F	F					
T	F	T					
T	T	F					
T	T	T					

Analyzing the Garfield Sentence with a Truth Table

q	r	s	$\neg s$	$r \vee \neg s$	$r \wedge s$	$(r \wedge s) \rightarrow q$	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	T	F	T	T	F	T	T
F	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
T	F	T	F	F	F	T	F
T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T

Converse, Contrapositive

Implication:

$$p \rightarrow r$$

Contrapositive:

$$\neg r \rightarrow \neg p$$

Converse:

$$r \rightarrow p$$

Inverse:

$$\neg p \rightarrow \neg r$$

Consider

p: *x* is divisible by 2

r: *x* is divisible by 4

$p \rightarrow r$	
$r \rightarrow p$	
$\neg r \rightarrow \neg p$	
$\neg p \rightarrow \neg r$	

Converse, Contrapositive

Implication:

$$p \rightarrow r$$

Converse:

$$r \rightarrow p$$

Contrapositive:

$$\neg r \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg r$$

<u>Consider</u>

p: ***x*** **is divisible by 2**

r**:** ***x*** **is divisible by 4**

$p \rightarrow r$	
$r \rightarrow p$	
$\neg r \rightarrow \neg p$	
$\neg p \rightarrow \neg r$	

	Divisible By 2	Not Divisible By 2
Divisible By 4		
Not Divisible By 4		

Converse, Contrapositive

Implication:

$p \rightarrow r$

Converse:

$r \rightarrow p$

Contrapositive:

$\neg r \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg r$

<u>Consider</u>

***p*: *x* is divisible by 2**

***r*: *x* is divisible by 4**

$p \rightarrow r$	
$r \rightarrow p$	
$\neg r \rightarrow \neg p$	
$\neg p \rightarrow \neg r$	

	Divisible By 2	Not Divisible By 2
Divisible By 4	4,8,12,...	Impossible
Not Divisible By 4	2,6,10,...	1,3,5,...

Converse, Contrapositive

Implication:

$p \rightarrow r$

Converse:

$r \rightarrow p$

Contrapositive:

$\neg r \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg r$

<u>Consider</u>

p: *x* is divisible by 2

r: *x* is divisible by 4

$p \rightarrow r$	**F**
$r \rightarrow p$	**T**
$\neg r \rightarrow \neg p$	**F**
$\neg p \rightarrow \neg r$	**T**

	Divisible By 2	Not Divisible By 2
Divisible By 4	4,8,12,...	Impossible
Not Divisible By 4	2,6,10,...	1,3,5,...

Converse, Contrapositive

Implication:

$p \rightarrow r$

Contrapositive:

$\neg r \rightarrow \neg p$

Converse:

$r \rightarrow p$

Inverse:

$\neg p \rightarrow \neg r$

How do these relate to each other?

p	r	$p \rightarrow r$	$r \rightarrow p$	$\neg p$	$\neg r$	$\neg p \rightarrow \neg r$	$\neg r \rightarrow \neg p$
T	T						
T	F						
F	T						
F	F						

Converse, Contrapositive

Implication:

$$p \rightarrow r$$

Converse:

$$r \rightarrow p$$

Contrapositive:

$$\neg r \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg r$$

An implication and it's contrapositive have the same truth value!

p	r	$p \rightarrow r$	$r \rightarrow p$	$\neg p$	$\neg r$	$\neg p \rightarrow \neg r$	$\neg r \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Application: Digital Circuits

Computing With Logic

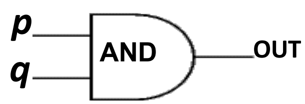
- T corresponds to 1 or "high" voltage
- F corresponds to 0 or "low" voltage

Gates

- Take inputs and produce outputs (functions)
- Several kinds of gates
- Correspond to propositional connectives (most of them)

Last class: AND, OR, NOT Gates

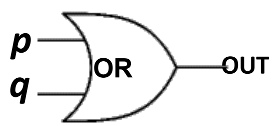
AND Gate



p	*q*	OUT
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

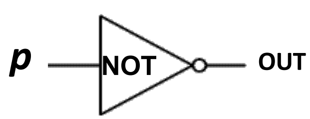
OR Gate



p	*q*	OUT
1	1	1
1	0	1
0	1	1
0	0	0

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

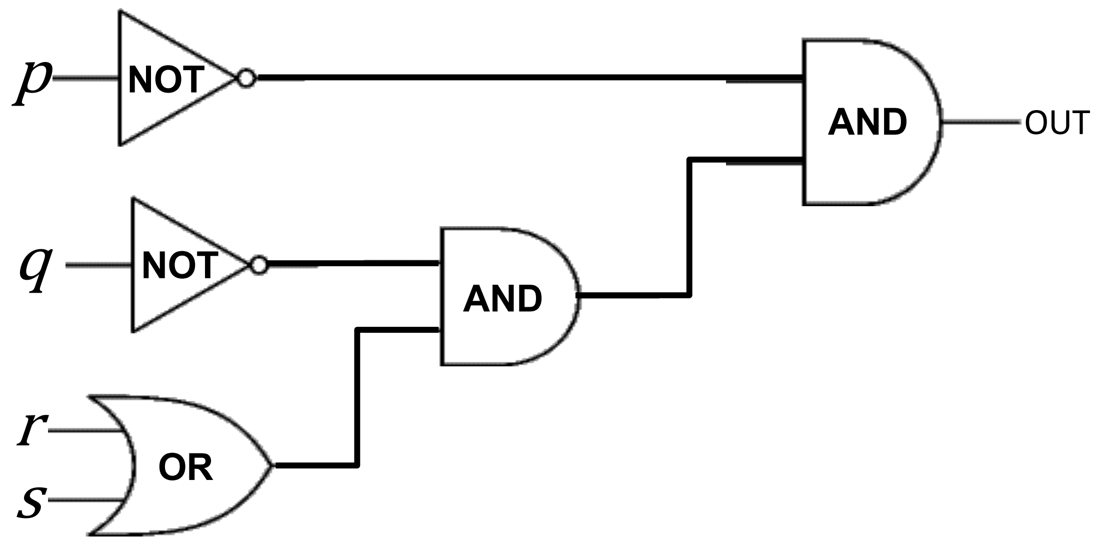
NOT Gate



p	OUT
1	0
0	1

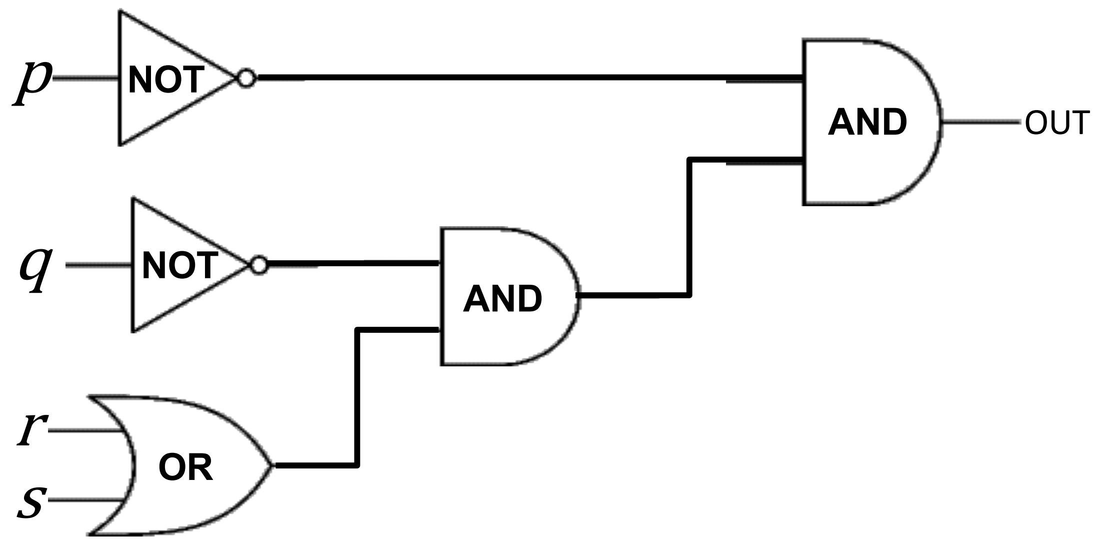
p	$\neg p$
T	F
F	T

Combinational Logic Circuits



Values get sent along wires connecting gates

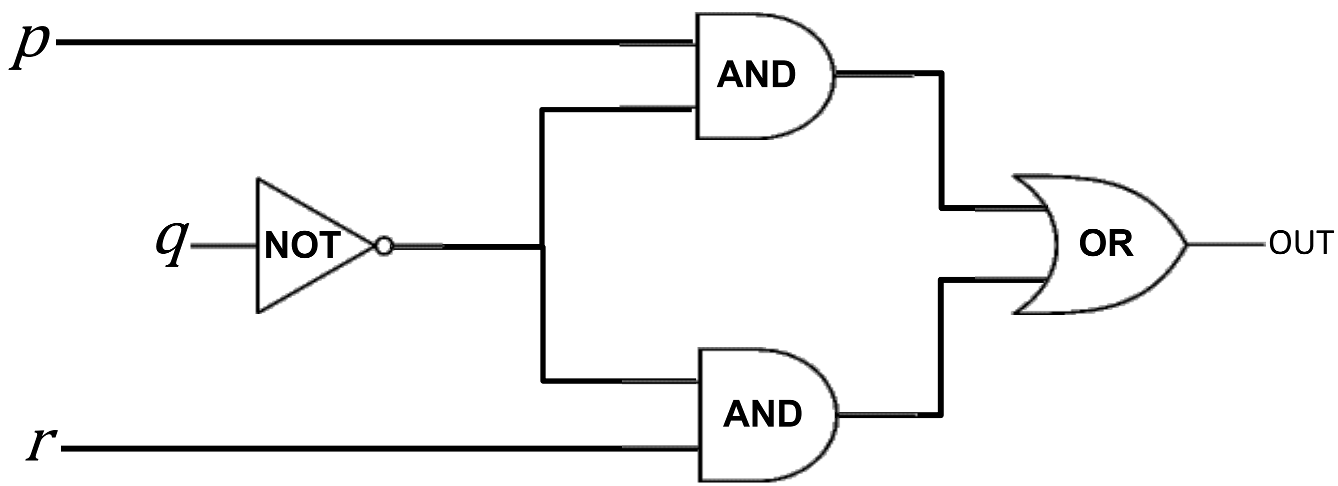
Combinational Logic Circuits



Values get sent along wires connecting gates

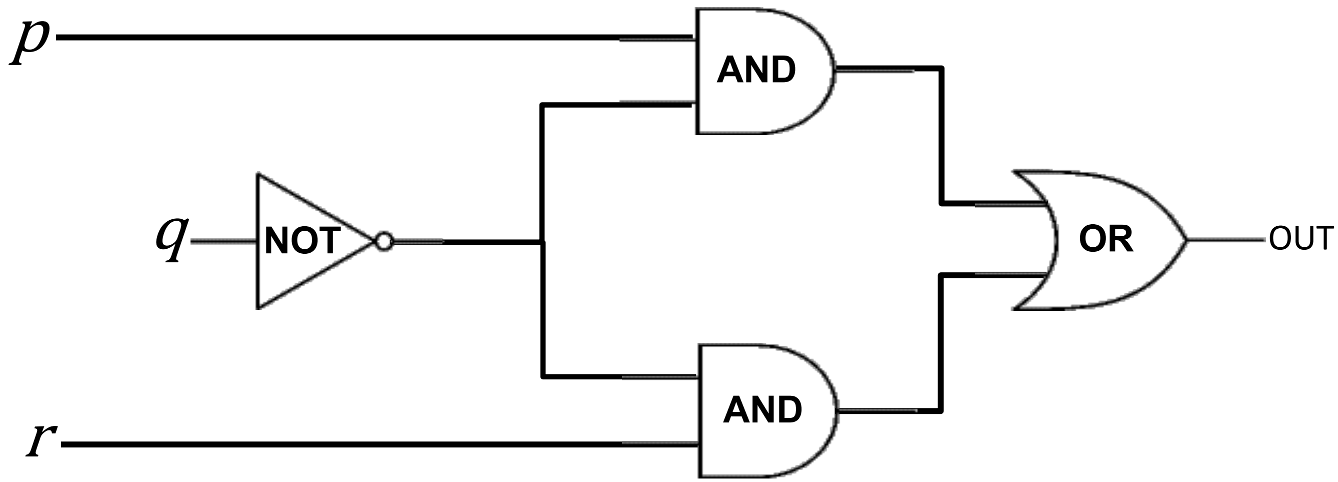
$$\neg p \wedge (\neg q \wedge (r \vee s))$$

Combinational Logic Circuits



Wires can send one value to multiple gates!

Combinational Logic Circuits



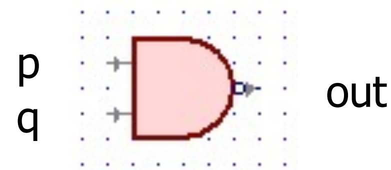
Wires can send one value to multiple gates!

$$(p \wedge \neg q) \vee (\neg q \wedge r)$$

Other Useful Gates

NAND

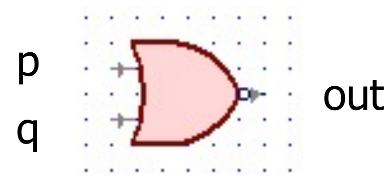
$\neg(p \wedge q)$



p	q	out
0	0	1
0	1	1
1	0	1
1	1	0

NOR

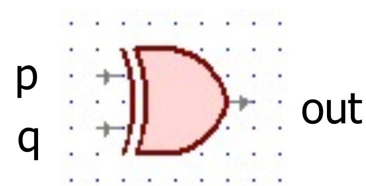
$\neg(p \vee q)$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	0

XOR

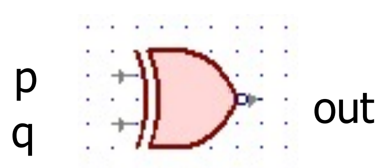
$p \oplus q$



p	q	out
0	0	0
0	1	1
1	0	1
1	1	0

XNOR

$p \leftrightarrow q$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	1

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

$p \oplus p$

$(p \rightarrow r) \wedge p$

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

This is a tautology. It's called the "law of the excluded middle".
If p is true, then $p \vee \neg p$ is true. If p is false, then $p \vee \neg p$ is true.

$p \oplus p$

This is a contradiction. It's always false no matter what truth value p takes on.

$(p \rightarrow r) \wedge p$

This is a contingency. When p=T, r=T, $(T \rightarrow T) \wedge T$ is true.
When p=T, r=F, $(T \rightarrow F) \wedge T$ is false.

Logical Equivalence

A = B means A and B are identical “strings”:

- $p \wedge r = p \wedge r$

- $p \wedge r \neq r \wedge p$

Logical Equivalence

A = B means A and B are identical "strings":

- $p \wedge r = p \wedge r$

  These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

$A \equiv B$ means A and B have identical truth values:

- $p \wedge r \equiv p \wedge r$
- $p \wedge r \equiv r \wedge p$
- $p \wedge r \not\equiv r \vee p$

Logical Equivalence

A = B means A and B are identical "strings":

– $p \wedge r = p \wedge r$

These are equal, because they are character-for-character identical.

– $p \wedge r \neq r \wedge p$

These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means A and B have identical truth values:

– $p \wedge r \equiv p \wedge r$

Two formulas that are **equal** also are equivalent.

– $p \wedge r \equiv r \wedge p$

These two formulas have the same truth table!

– $p \wedge r \not\equiv r \vee p$

When p=T and r=F, $p \wedge r$ is false, but $p \vee r$ is true!

$A \leftrightarrow B$ vs. $A \equiv B$

$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of A and B.

$A \equiv B$ is an **assertion** over all possible truth values that A and B always have the same truth values.

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ have the same meaning.