Warm up

I have 4 cent stamps and 5 cent stamps (as many as I want of each).
We'll prove that I can make **exactly** n cents of stamps for all $n \geq 12$.

Try for a few values. Can you make $12, 13, 14$? What about 11? 10?

Even More Induction

CSE 311 Autumn 20
Lecture 17

Announcements

HW5 due date delayed until Wednesday.

- Affects our ability to get feedback to you before midterm ends.

Today is just more induction practice.

Monday is another kind of induction (structural induction)

- No problem on the midterm will require structural induction (but if you figure out how to use it, we'll take it). Slide deck from Wednesday (including stuff repeated today) is fair game.

More midterm details coming soon.

Induction on Primes

Let $P(i)$ be "i can be written as a product of primes."

We show $P(n)$ for all $n \geq 2$ by induction on n.

Base Case ($\boldsymbol{n = 2}$): 2 is a product of just itself. Since 2 is prime, it is written as a product of primes.

Inductive Hypothesis: Suppose $P(2), \ldots, P(k)$ hold for an arbitrary integer $k \geq 2$.

Inductive Step:
Case 1, $k + 1$ is prime: then $k + 1$ is automatically written as a product of primes.

Case 2, $k + 1$ is composite: We can write $k + 1 = st$ for s, t nontrivial divisors (i.e. $2 \leq s < k + 1$ and $2 \leq t < k + 1$). By inductive hypothesis, we can write s as a product of primes $p_1 \cdot \ldots p_j$ and t as a product of primes $q_1 \cdots q_\ell$. Multiplying these representations, $k + 1 = p_1 \cdots p_j \cdot q_1 \cdots q_\ell$, which is a product of primes.

Therefore $P(k + 1)$.

$P(n)$ holds for all $n \geq 2$ by the principle of induction.

Strong Induction

That hypothesis where we assume $P(\text{base case}), \ldots, P(k)$ instead of just $P(k)$ is called a strong inductive hypothesis.

Strong induction is the same fundamental idea as weak ("regular") induction.

$P(0)$ is true.
And $P(0) \to P(1)$, so $P(1)$.
And $P(1) \to P(2)$, so $P(2)$.
And $P(2) \to P(3)$, so $P(3)$.
And $P(3) \to P(4)$, so $P(4)$.
…

$P(0)$ is true.
And $P(0) \to P(1)$, so $P(1)$.
And $[\mathrm{P}(0) \land P(1)] \to P(2)$, so $P(2)$.
And $[\mathrm{P}(0) \land \cdots \land P(2)] \to P(3)$, so $P(3)$.
And$[\mathrm{P}(0) \land \cdots \land P(3)] \to P(4)$, so $P(4)$.
…

Making Induction Proofs Pretty

All of our strong induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.
2. Base Case: Show $P(b)$ i.e. show the base case
3. Inductive Hypothesis: Suppose $\mathrm{P(b)} \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq b$.
4. Inductive Step: Show $P(k+1)$ (i.e. get $[\mathrm{P(b)} \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$)
5. Conclude by saying $P(n)$ is true for all $n \geq b$ by the principle of induction.

Strong Induction vs. Weak Induction

Think of strong induction as “my recursive call might be on LOTS of smaller values” (like mergesort – you cut your array in half)

Think of weak induction as “my recursive call is always on one step smaller.”

Practical advice:

A strong hypothesis isn’t wrong when you only need a weak one (but a weak one is wrong when you need a strong one). Some people just always write strong hypotheses. But it’s easier to typo a strong hypothesis.

Robbie leaves a blank spot where the IH is, and fills it in after the step.

Let’s Try Another! Stamp Collecting

I have 4 cent stamps and 5 cent stamps (as many as I want of each).
Prove that I can make exactly n cents worth of stamps for all $n \geq 12$.

Try for a few values.

Then think…how would the inductive step go?

Stamp Collection (attempt)

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose [maybe some other stuff and] $P(k)$, for an arbitrary $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k - 3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k + 1$ cents.

Stamp Collection

Is the proof right?

How do we know $P(13)$

We're not the base case, so our inductive hypothesis assumes $P(12)$, and then we say if $P(9)$ then $P(13)$.

Wait a second….

If you go back s steps every time, you need s base cases.

Or else the first few values aren't proven.

Stamp Collection

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.
13 cents can be made with two 4 cent stamps and one 5 cent stamp.
14 cents can be made with one 4 cent stamp and two 5 cent stamps.
15 cents can be made with three 5 cent stamps.

Inductive Hypothesis Suppose $\mathrm{P}(12) \wedge P(13) \wedge \cdots \wedge P(k)$, for an arbitrary $k \geq 15$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k - 3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k + 1$ cents.

A good last check

After you've finished writing an inductive proof, pause.

If your inductive step always goes back s steps, you need s base cases (otherwise $b + 1$ will go back before the base cases you've shown). And make sure your inductive hypothesis is strong enough.

If your inductive step is going back a varying (unknown) number of steps, check the first few values above the base case, make sure your cases are really covered. And make sure your IH is strong.

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.

2. Base Cases: Show $P(b_{min}), P(b_{min+1}) \ldots P(b_{max})$ i.e. show the base cases

3. Inductive Hypothesis: Suppose $P(b_{min}) \land P(b_{min}+1) \land \cdots \land P(k)$ for an arbitrary $k \geq b_{max}$. (The smallest value of k assumes **all** bases cases, but nothing else)

4. Inductive Step: Show $P(k+1)$ (i.e. get $[\mathrm{P}(\mathrm{b}_{min}) \land \cdots \land P(k)] \rightarrow P(k+1)$)

5. Conclude by saying $P(n)$ is true for all $n \geq b_{min}$ by the principle of induction.

Practical Advice

How many base cases do you need?

Always at least one.

If you're analyzing recursive code or a recursive function, at least one for each base case of the code/function.

If you always go back s steps, at least s consecutive base cases.

Enough to make sure every case is handled.

Stamp Collection, Done Wrong

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.
Inductive Hypothesis Suppose $P(k)$, $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make k cents exactly with stamps. Replace one of the 4 cent stamps with a 5 cent stamp.

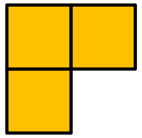
$P(n)$ holds for all n by the principle of induction.

Stamp Collection, Done Wrong

What if the starting point doesn’t have any 4 cent stamps?

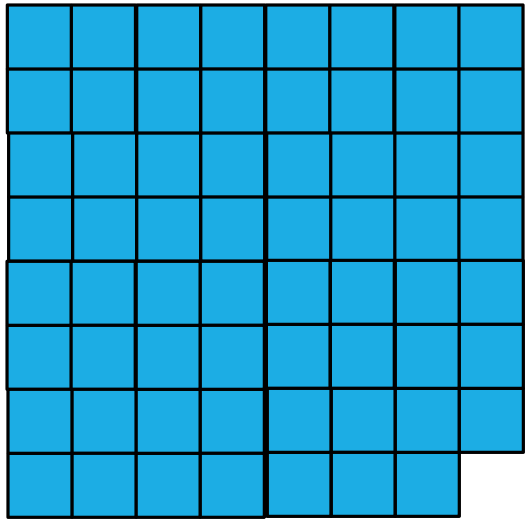
Like, say, 15 cents = 5+5+5.

Gridding



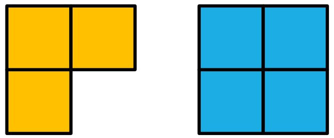
I've got a bunch of these 3 piece tiles.

I want to fill a 2^nx2^n grid ($n \geq 1$) with the pieces, except for a 1x1 spot in a corner.



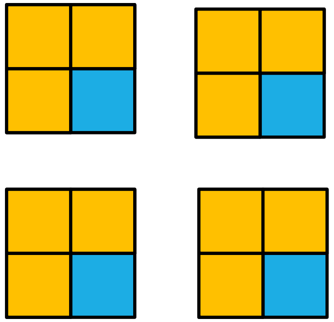
Gridding: Not a formal proof, just a sketch

Base Case: $n = 1$



Inductive hypothesis: Suppose you can tile a 2^kx2^k grid, except for a corner.

Inductive step: 2^{k+1}x2^{k+1}, divide into quarters. By IH can tile…



Recursively Defined Functions

Just like induction works will with recursive code, it also works well for recursively-defined functions.

Define the Fibonacci numbers as follows:

$f(0) = 1$

$f(1) = 1$

$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

*This is a somewhat unusual definition, $f(0) = 0, f(1) = 1$ is more common.

Fibonacci Inequality

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

$$f(0) = 1; \qquad f(1) = 1$$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Fibonacci Inequality

$f(0) = 1; \qquad f(1) = 1$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be "$f(n) \leq 2^n$" We show $P(n)$ is true for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0)$: $f(0) = 1 \leq 1 = 2^0$.

$(n = 1)$: $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \land P(1) \land \cdots \land P(k)$ for an arbitrary $k \geq 1$.

Inductive step:

Target: $P(k+1)$. i.e. $f(k+1) \leq 2^{k+1}$

Fibonacci Inequality

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be "$f(n) \leq 2^n$" We show $P(n)$ is true for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0)$: $f(0) = 1 \leq 1 = 2^0$.

$(n = 1)$: $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 1$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have $f(k+1) \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$.

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

Claim: $3 | (2^{2n} - 1)$ for all $n \in \mathbb{N}$.

[Define $P(n)$]

Base Case

Inductive Hypothesis

Inductive Step

[conclusion]

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step:

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$

FORCE the expression in your IH to appear

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4\left(2^{2k} - 1\right) + 4 - 1$

By IH, we can replace $2^{2k} - 1$ with $3j$ for an integer j

$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$

Since $4j + 1$ is an integer, we meet the definition of divides and we have:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$2^{2\cdot 0} - 1 = 0 = 3 \cdot 0$

$2^{2\cdot 1} - 1 = 3 = 3 \cdot 1$

$2^{2\cdot 2} - 1 = 15 = 3 \cdot 5$

$2^{2\cdot 3} - 1 = 63 = 3 \cdot 21$

$2^{2\cdot 4} - 1 = 255 = 3 \cdot 85$

$2^{2\cdot 5} - 1 = 1023 = 3 \cdot 341$

The divisor goes from k to $4k + 1$

$$0 \to 4 \cdot 0 + 1 = 1$$
$$1 \to 4 \cdot 1 + 1 = 5$$
$$5 \to 4 \cdot 5 + 1 = 21$$

…

That might give us a hint that $4k + 1$ will be in the algebra somewhere, and give us another intermediate target.

Induction: Hats!

You have n people in a line ($n \geq 2$). Each of them wears either a purple hat or a gold hat. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that there is a person with a purple hat next to someone with a gold hat.

Yes this is kinda obvious. I promise this is good induction practice.

Yes you could argue this by contradiction. I promise this is good induction practice.

Induction: Hats!

Define $P(n)$ to be "in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider a line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider a line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2:. There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length k, has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Fibonacci Inequality Two

$f(0) = 1; \quad f(1) = 1$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

[Define $P(n)$]

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

Fibonacci Inequality Two

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

Target: $f(k+1) \geq 2^{(k+1)/2}$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

Fibonacci Inequality Two

$f(0) = 1; \qquad f(1) = 1$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$f(k+1) \geq 2^{k/2} + 2^{(k-1)/2}$

$\geq 2^{(k+1)/2}$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

Fibonacci Inequality Two

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$$\begin{aligned} f(k+1) &\geq 2^{k/2} + 2^{(k-1)/2} \\ &= 2^{(k-1)/2}\left(\sqrt{2} + 1\right) \\ &\geq 2^{(k-1)/2} \cdot 2 \\ &\geq 2^{(k+1)/2} \end{aligned}$$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

More Practice

Even More Induction Practice

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Claim: $h(n) \geq g(n)$ for all integers $n \geq 1$

Even More Induction Practice

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case

Inductive Hypothesis:

Inductive Step:

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$g(k+1) = (k+1) \cdot g(k)$

$= (k+1)^{k+1}$.

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$\begin{aligned} g(k+1) &= (k+1) \cdot g(k) \\ &\leq (k+1) \cdot h(k) \text{ by IH.} \\ \\ &= (k+1)^{k+1}. \end{aligned}$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$
\begin{aligned}
g(k+1) &= (k+1) \cdot g(k) \\
&\leq (k+1) \cdot h(k) && \text{by IH.} \\
&\leq (k+1) \cdot k^k && \text{by definition of } h(k) \\
&\leq (k+1) \cdot (k+1)^k \\
&= (k+1)^{k+1}.
\end{aligned}
$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$\begin{aligned} g(k+1) &= (k+1) \cdot g(k) \\ &\leq (k+1) \cdot h(k) && \text{by IH.} \\ &\leq (k+1) \cdot k^k && \text{by definition of } h(k) \\ &\leq (k+1) \cdot (k+1)^k \\ &= (k+1)^{k+1}. \end{aligned}$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$):

Inductive Hypothesis:

Inductive Step:

[Conclusion]

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

Target: $\sum_{i=0}^{k+1} 2 + 3i = \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^{k} 2 + 3i) + \left(2 + 3(k+1)\right)$. By IH, we have:

$\sum_{i=0}^{k+1} 2 + 3i = \frac{(\mathrm{k}+1)(3\mathrm{k}+4)}{2} + 2 + 3\mathrm{k} + 3 = ????$

$$= \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2+3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n=0$): $\sum_{i=0}^{0} 2+3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2+3i = \left(\sum_{i=0}^{k} 2+3i\right) + \left(2+3(k+1)\right)$. By IH, we have:

$\sum_{i=0}^{k+1} 2+3i = \frac{(\mathrm{k}+1)(3\mathrm{k}+4)}{2} + 2 + 3\mathrm{k} + 3 = \frac{3\mathrm{k}^2+7\mathrm{k}+4}{2} + \frac{6\mathrm{k}+10}{2} = \frac{3\mathrm{k}^2+13\mathrm{k}+14}{2} = \frac{(3\mathrm{k}+7)(\mathrm{k}+2)}{2} = \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.