Plan for Today

• Strategic behavior related to transaction fees in Bitcoin

• Revisiting basics of equilibria

• Best response dynamics and potential games

• Online learning as a way to play games
Transaction fees in Bitcoin

Q: What will happen when block rewards are negligible & all revenue is transaction fees.

Model
- Transactions arrive at constant rate
- In an interval of units of time, the total amount of fees is \( C \cdot T \) (\( C=1 \))
- Blocks are created at constant rate
- \( R \) transaction fees available
- Miners can put any fraction into block
- Miners have enough space to include everything.

Strategic decisions
- Which block to extend?
- How much of outstanding transactions to include
- When to publish found blocks

Protocol (honest miners)
- Mine on longest chain (tie breaking for what heard about first)
- Include all transactions you know about
- Publish found block immediately

Petyr Compliant Strategy

\[
\begin{array}{c}
\text{PB5} \\
\hline
10
\end{array}
\]

break ties for longest chain w/ most leftover fees
Lazy underwriting. Suppose $a > b$.

$$F = \begin{cases} 
100 & \text{if } a \leq b \\
55 & \text{if } a > b 
\end{cases}$$

$F(\frac{a+b}{2}) = \frac{a+b}{2}$

**Function Fork $(f)$**

$$f(x) = kx \quad 0 < k < 1$$
**Theorem 5.1.** For any constant \( y \leq 1/2 \) such that \( 2y - \ln(y) \geq 2 \), define:

\[
f(x) = x, \quad \forall \ x \leq y \quad (1)
\]

\[
f(x) = -W_0(-ye^{x-2y}), \quad \forall \ y < x < 2y - \ln(y) - 1 \quad (2)
\]

\[
f(x) = 1, \quad \forall \ x \geq 2y - \ln(y) - 1 \quad (3)
\]

Then it is an equilibrium for every miner to use the strategy \textsc{Function-fork}(f) as long as:

- Every miner is non-atomic.
- Miners may only mine on top of chains of length \( H \) or \( H - 1 \).

Furthermore, in any such equilibrium, the expected number of backlogged transactions after \( n \) time steps is \( \Theta(\sqrt{n}) \).

\(^5\)Such \( y \) exist. This range is \((0, \approx 0.2]\).
Equilibria

Cost-minimization games.
- $k$ players
- $A_i$: strategies of player $i$
- $c_i(s_i)$

$ar{s}$ is a pure NE if $\forall i, \forall s_i \in A_i$

$$c_i(s_i^i, \bar{s}_-i) \geq c_i(s_i, \bar{s}_-i)$$

$p^\pi=(p_1, ..., p_k)$ are prob dists where $p_i$ is a prob dist over $A_i$.
$p^\pi$ is a mixed NE if $\forall i, \forall s_i$

$$E_{\bar{s} \sim p^\pi} \left[ c_i(s_i^i, s_-i) \right] \geq E_{\bar{s} \sim p} \left[ c_i(s_i^\pi) \right]$$
Any finite game has a mixed NE.

Correlated equilibrium.

Dism $\bar{p}$ on $A_1 \times A_2 \times \ldots \times A_k$
over strategy profiles
is a correlated equilibrium if $\forall i,$
$\forall s_i, s'_i \in A_i$:

$$E_{s \sim \bar{p}} [c_i(s_i) | s_i] \leq E_{s \sim \bar{p}} [c_i(s_i', s_{-i}) | s_i]$$

Traffic Light

<table>
<thead>
<tr>
<th></th>
<th>W/S</th>
<th>E/N</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step</strong></td>
<td><strong>Green</strong></td>
<td><strong>Red</strong></td>
</tr>
<tr>
<td>Step</td>
<td>$\frac{1}{3} (9,0)$</td>
<td>$\frac{2}{3} (0,1)$</td>
</tr>
<tr>
<td>Go</td>
<td>$\frac{2}{3} (1,0)$</td>
<td>$-9,5$</td>
</tr>
</tbody>
</table>
The distribution $\pi$ on $A_1 \times A_2 \times \ldots \times A_n$ is a coarse correlated equilibrium (CCE) if for all $i$, $s_i$, $s'_i$:

$$E_{\pi} \left[ c_i(s) \right] \leq E_{\pi} \left[ c_i(s_i, s_{-i}) \right] + \epsilon$$

Best response dynamics:
- If this halts $\rightarrow$ pure NE.
- Repeatedly to reach pure NE even if one exists.

Potential games:
- $f_i : T(A_i) \rightarrow \mathbb{R}$
- $\forall s \in (a_1, a_2, \ldots) \forall i, s_i \in A_i$
- $c_i(s_i, s_{-i}) - c_i(s) = f_i(s_i, s_{-i}) - f_i(s)$

Every potential game has a pure NE. Consider minimizing $f$. 
Network formation games

\[ p = (p_1, p_2, \ldots, p_k) \]

\[ \phi(p^*) = \sum_{e \in E(p^*)} \sum_{k=1}^{n_e(p^*)} \frac{ce}{\kappa} \]

\[ n_e(p^*) - \text{# paths in edge } e. \]

\[ c_i(s; s; i) - c_i(s^*) = \phi(s; s; i) - \phi(s^*) \]
Online learning single player game against adversary

A: set of possible actions player can take each day |A| = n

for $t = 1, \ldots, T$
player picks $a_t \in A$
prob distn over actions
$p_1, p_2, \ldots, p_n$

$c_i \in [0,1]
\sum_i c_i = 1$

Expected cost on day $T$

$\sum_{i=1}^{n} p_i c_i$

$\frac{p^*}{c^*}$

Benchmark?

$\sum_{t=1}^{T} \min_{a_t \in A} c^+(a_t)$

Impossible.

Ex 2 acts.

Every day adv picks
1 path cost 1
other cost 0.

Exp cost of any alg $\frac{T}{2}$
The regret of an alg on a sequence of cost vectors:

\[
\frac{1}{T} \sum_{t=1}^{T} c_t(a^*) - \min_{a \in A} \sum_{t=1}^{T} c_t(a)
\]

\[
\sum_{a \in A} p^t(a) c_t(a)
\]

Goal: get regret \( \frac{0}{T} \rightarrow 0 \)

\[\text{play act}_3 \rightarrow \text{cost 1} \rightarrow \text{act 3}
\]

\[\Rightarrow \text{alg pay } T \]

3 arm that has cost \( \leq \frac{T}{n} \)

\( n \) is #arms

\[
\text{Best we can hope for is regret } \sqrt{\frac{2 \ln n}{T}}
\]

\[
\text{Adv} \quad n=2
\]

\[
\begin{pmatrix}
(1, 0) \\
(0, 1)
\end{pmatrix}
\]

Any alg has expected \( \frac{T}{2} \)

\[
\mathbb{E} \left[ \min_{i \in [n]} \text{in } T 	ext{ tosses} \right] = \frac{T}{2} - \sqrt{\frac{T}{2}}
\]
For each action maintain weight \( w^+ = (w^+_1, w^+_2, \ldots, w^+_N) \)

Initialize \( w^+(a) = 1 \) \( \forall a \)

For \( t = 1 \) to \( T \)

Choose \( a^+ \) with prob proportional to \( w^+ \)

\( \rho^+(a) = \frac{w^+(a)}{\sum_{a \in A} w^+(a)} \)

given \( c^+ \)

\( w^+(a) = w^+(a) \left( 1 - \epsilon \right)^{c^+(a)} \)

\( \epsilon \) parameter \((0, 1)\) to be set.

Then this alg has rate \( \tilde{O}(N/\epsilon) \)

\[ \text{let } s_i \text{ be the uniform distr on } s_i^1, s_i^2, \ldots, s_i^T \]

Let \( \tilde{p} \) be the uniform distr

\( \tilde{p} \) is approx CCE

The regret of player \( i \) at end of \( T \) rounds

\[ \text{regret of player } i \text{ at end of } T \text{ rounds} = \frac{1}{T} \left[ \sum_{t=1}^{T} c_i (s_t^i) - \min_{s_i \in S_i} \sum_{t=1}^{T} c_i (s_t^i, s_{-i}^t) \right] \]