CSEP 590
Data Compression
Autumn 2007

Lossy Image Compression
Transform Coding
JPEG

Lossy Image Compression Methods

- DCT Compression
  - JPEG
- Scalar quantization (SQ).
- Vector quantization (VQ).
- Wavelet Compression
  - SPIHT
  - UWIC (University of Washington Image Coder)
  - EBCOT
  - JPEG 2000

JPEG Standard

- JPEG - Joint Photographic Experts Group
  - Current image compression standard. Uses discrete cosine transform, scalar quantization, and Huffman coding.
- JPEG 2000 uses to wavelet compression.

Barbara

32:1 compression ratio
.25 bits/pixel (8 bits)

JPEG

VQ

SPIHT

Original

Images and the Eye

- Images are meant to be viewed by the human eye (usually).
- The eye is very good at “interpolation”, that is, the eye can tolerate some distortion. So lossy compression is not necessarily bad. The eye has more acuity for luminance (gray scale) than chrominance (color).
  - Gray scale is more important than color.
  - Compression is usually done in the YUV color coordinates, Y for luminance and U,V for color.
  - U and V should be compressed more than Y
  - This is why we will concentrate on compressing gray scale (8 bits per pixel) images.

Distortion

original → x → Encoder → compressed y → Decoder → decompressed $\hat{x}$

- Lossy compression: $x \neq \hat{x}$
- Measure of distortion is commonly mean squared error (MSE). Assume x has n real components (pixels).

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{x}_i)^2$$

PSNR

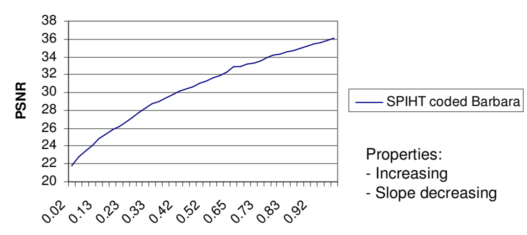
- Peak Signal to Noise Ratio (PSNR) is the standard way to measure fidelity.

$$PSNR = 10\log_{10}\left(\frac{m^2}{MSE}\right)$$

where m is the maximum value of a pixel possible.
For gray scale images (8 bits per pixel) m = 255.

- PSNR is measured in decibels (dB).
  - .5 to 1 dB is said to be a perceptible difference.
  - Decent images start at about 30 dB

Rate-Fidelity Curve



Properties:
- Increasing
- Slope decreasing

PSNR is not Everything

VQ

PSNR = 25.8 dB

PSNR = 25.8 dB

PSNR Reflects Fidelity (1)

VQ

PSNR 25.8
.63 bpp
12.8 : 1

PSNR Reflects Fidelity (2)

VQ

PSNR 24.2
.31 bpp
25.6 : 1

PSNR Reflects Fidelity (3)

VQ

PSNR 23.2
.16 bpp
51.2 : 1

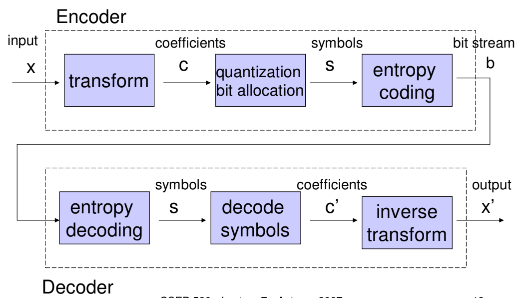
Idea of Transform Coding

- Transform the input pixels $x_0,x_2,\ldots,x_{N-1}$ into coefficients $c_0,c_1,\ldots,c_{N-1}$ (real values)
  - The coefficients have the property that most of them are near zero
  - Most of the “energy” is compacted into a few coefficients
- Quantize the coefficients
  - This is where there is loss, since coefficients are only approximated
  - Important coefficients are kept at higher precision
- Entropy encode the quantization symbols

Decoding

- Entropy decode the quantized symbols
- Compute approximate coefficients $c'_0,c'_1,\ldots,c'_{N-1}$ from the quantized symbols.
- Inverse transform $c'_0,c'_1,\ldots,c'_{N-1}$ to $x'_0,x'_1,\ldots,x'_{N-1}$ which is a good approximation of the original $x_0,x_2,\ldots,x_{N-1}$.

Block Diagram of Transform Coding



Mathematical Properties of Transforms

- Linear Transformation - Defined by a real nxn matrix $A = (a_{ij})$

$$\begin{bmatrix} a_{00} & \cdots & a_{0,N-1} \\ \vdots & & \vdots \\ a_{N-1,0} & \cdots & a_{N-1,N-1} \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

- Orthonormality $A^{-1} = A^T$ (transpose)

Why Coefficients

$$A^T c = x$$

$$\begin{bmatrix} a_{00} & \cdots & a_{N-1,0} \\ \vdots & & \vdots \\ a_{0,N-1} & \cdots & a_{N-1,N-1} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$$\begin{bmatrix} a_{00} \\ \vdots \\ a_{0,N-1} \end{bmatrix} c_0 + \cdots + \begin{bmatrix} a_{N-1,0} \\ \vdots \\ a_{N-1,N-1} \end{bmatrix} c_{N-1} = \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

basis vectors

coefficients

Why Orthonomality

- The energy of the data equals the energy of the coefficients

$$\sum_{i=0}^{N-1} c_i^2 = c^T c = (Ax)^T (Ax)$$

$$= (x^T A^T)(Ax) = x^T (A^T A) x = x^T x = \sum_{i=0}^{N-1} x_i^2$$

Squared Error is Preserved with Orthonormal Transformations

- In lossy coding we only send an approximation c'_i of c_i because it takes fewer bits to transmit the approximation.
  Let $c_i = c'_i + \varepsilon_i$

$$\sum_{i=0}^{N-1} \varepsilon_i^2 = \sum_{i=0}^{N-1} (c_i - c'_i)^2 = (c - c')^T (c - c') = (Ax - Ax')^T (Ax - Ax')$$

$$= (A(x - x'))^T (A(x - x')) = ((x - x')^T A^T)(A(x - x'))$$

$$= (x - x')^T (A^T A)(x - x') = (x - x')^T (x - x')$$

$$= \sum_{i=0}^{N-1} (x_i - x'_i)^2$$ Squared error in original.

Compaction Example

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^2 = A \Rightarrow A^{-1} = A$$

$$A^T = A = A^{-1}$$ orthonormal

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{2} b \\ 0 \end{bmatrix}$$ compaction

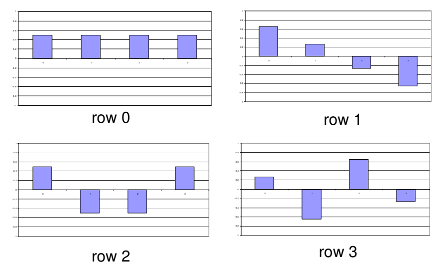
Discrete Cosine Transform

$$d_{ij} = \begin{cases} \sqrt{\dfrac{1}{N}} & \text{if } i = 0 \\ \sqrt{\dfrac{2}{N}}\cos\dfrac{(2j+1)i\pi}{2N} & \text{if } i > 0 \end{cases}$$

N = 4

$$D = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .65328 & .270598 & -.270598 & -.65328 \\ .5 & -.5 & -.5 & .5 \\ .270598 & -.65328 & .65328 & -.270598 \end{bmatrix}$$

Basis Vectors

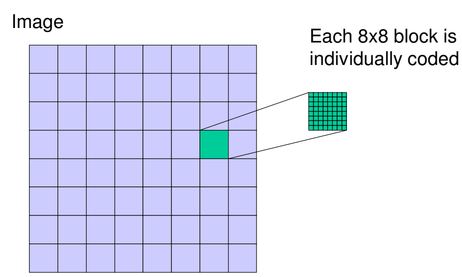


Decomposition in Terms of Basis Vectors

$$\begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix} c_0 + \begin{bmatrix} .653281 \\ .270598 \\ -.270598 \\ -.653281 \end{bmatrix} c_1 + \begin{bmatrix} .5 \\ -.5 \\ -.5 \\ .5 \end{bmatrix} c_2 + \begin{bmatrix} .270598 \\ -.653281 \\ .653281 \\ -.270598 \end{bmatrix} c_3 = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

DC coefficient AC coefficients

Block Transform



2-Dimensional Block Transform

Block of pixels X

x_{00}	x_{01}	x_{02}	x_{03}
x_{10}	x_{11}	x_{12}	x_{13}
x_{20}	x_{21}	x_{22}	x_{23}
x_{30}	x_{31}	x_{32}	x_{33}

Transform

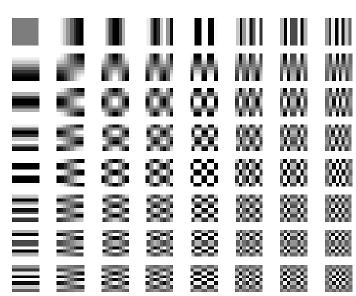
$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Transform rows $r_{ij} = \sum_{k=0}^{N-1} a_{kj} x_{ik}$

Transform columns $c_{ij} = \sum_{m=0}^{N-1} a_{im} r_{mj} = \sum_{m=0}^{N-1} a_{im} \sum_{k=0}^{N-1} a_{kj} x_{mk} = \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} a_{im} a_{kj} x_{mk}$

Summary $C = AXA^T$

8x8 DCT Basis



Importance of Coefficients

- The DC coefficient is the most important.
- The AC coefficients become less important as they are farther from the DC coefficient.
- Example Bit Allocation

8	7	5	3	2	1	0	0
7	5	3	2	1	0	0	0
5	3	2	1	0	0	0	0
3	2	1	0	0	0	0	0
2	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

compression
65 bits for 64
pixels = .86 bpp

Quantization

- For a nxn block we construct a nxn matrix Q such that Q_{ij} indicates how many quantization levels to use for coefficient c_{ij}.
- Encode c_{ij} with the label

$$s_{ij} = \left\lfloor \frac{c_{ij}}{Q_{ij}} + 0.5 \right\rfloor$$

Larger Q_{ij} indicates fewer levels.

- Decode s_{ij} to

$$c'_{ij} = s_{ij} Q_{ij}$$

Example Quantization

- c = 54.2, Q = 24 $\quad s = \left\lfloor \frac{54.2}{24} + 0.5 \right\rfloor = 2$

$c' = 2 \cdot 24 = 48$

- c = 54.2, Q = 12 $\quad s = \left\lfloor \frac{54.2}{12} + 0.5 \right\rfloor = 5$

$c' = 5 \cdot 12 = 60$

- c= 54.2, Q = 6 $\quad s = \left\lfloor \frac{54.2}{6} + 0.5 \right\rfloor = 9$

$c' = 9 \cdot 6 = 54$

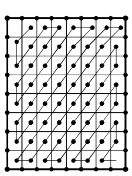
Example Quantization Table

16	11	10	16	24	40	51	61
12	12	14	19	26	58	60	55
14	13	16	24	40	57	69	56
14	17	22	29	51	87	80	62
18	33	37	56	68	109	103	77
24	35	55	64	81	104	113	92
49	64	78	87	103	121	120	101
72	92	95	98	112	100	103	99

Increase the bit rate = halve the table
Decrease the bit rate = double the table

Zig-Zag Coding

- DC label is coded separately.
- AC labels are usually coded in zig-zag order using a special entropy coding to take advantage of the ordering of the bit allocation (quantization).



JPEG (1987)

- Let $P = [p_{ij}]$, $0 \le i,j < N$ be an image with $0 \le p_{ij} < 256$.
- Center the pixels around zero
  - $x_{ij} = p_{ij} - 128$
- Code 8x8 blocks of P using DCT
- Choose a quantization table.
  - The table depends on the desired quality and is built into JPEG
- Quantize the coefficients according to the quantization table.
  - The quantization symbols can be positive or negative.
- Transmit the labels (in a coded way) for each block.

Block Transmission

- DC coefficient
  - DC coefficients don't change much from block to neighboring block. Hence, their labels change even less.
  - Predictive coding using differences is used to code the DC label.
- AC coefficients
  - Do a zig-zag coding.

Example Block of Labels

5	2	0	0	0	0	0	0
-8	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0
1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Coding order of AC labels
2 –8 3 0 0 0 0 1 1 0 0 1 0 0 .....

Coding Labels

- Categories of labels
  - 1 {0}
  - 2 {-1, 1}
  - 3 {-3,-2,2,3}
  - 4 {-7,-6,-5,-4,4,5 6 7}
- Label is indicated by two numbers C,B
- Examples

label	C,B
0	1
2	3, 2
-4	4, 3

Coding AC Label Sequence

- A symbol has three parts (Z,C,B)
  - Z for number of zeros preceding a label $0 \leq Z \leq 15$
  - C for the category of the of the label
  - B for a C-1 bit number for the actual label
- End of Block symbol (EOB) means the rest of the block is zeros. EOB = (0,0,-)
- Example: 2 –8 3 0 0 0 0 1 1 0 0 1 0 0 .....

(0,3,2)(0,5,7)(0,3,3)(4,2,1)(0,2,1)(2,2,1)(0,0,-)

Coding AC Label Sequence

- Z,C have a prefix code
- B is a C-1 bit number

Partial prefix code table

Z \ C	0	1	2	3
0	1010	00	01	100
1		1100	11011	11110001
2		1110	11111001	1111110111
3		111010	111110111	111111110101

(0,3,2) (0,5,7) (0,3,3) (4,2,1) (0,2,1) (2,2,1) (0,0,-)

100 10 11010 0111 100 11 11111111000 1 01 1 11111001 1 1010

46 bits representing 64 pixels = .72 bpp

Notes on Transform Coding

- Video Coding
  - MPEG – uses DCT
  - H.263, H.264 – uses DCT
- Audio Coding
  - MP3 = MPEG 1- Layer 3 uses DCT
- Alternative Transforms
  - Lapped transforms remove some of the blocking artifacts.
  - Wavelet transforms do not need to use blocks at all.