Quantum Computing Problem Set 4

Author: Dave Bacon (*Department of Computer Science & Engineering, University of Washington*)
Due: July 27, 2005

Problem 1: n Qubit Registers

In this problem we are given n qubits with the wave function

$$|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

.

(a) Suppose that we apply bit flips to all n qubits which have the wave function $|\phi\rangle$, i.e. we apply the unitary operation $X^{\otimes n}$ to this quantum system. Show that this does not change the wave function of the system.

(b) Suppose that we apply phase flips to all n qubits which have the wave function $|\phi\rangle$, i.e. we apply the unitary operation $Z^{\otimes n}$ to this quantum system. The new wave function will be of the form

$$|\phi'\rangle = \sum_{x=0}^{2^n-1} a_x |x\rangle.$$

Express a_x in terms of the bitstring x (in other words, a_x should be a function of $x_1, x_2, \dots, x_n$.)

(c) Suppose that we apply a bit flip to the last of n qubits which have the wave function $|\phi\rangle$, i.e. we apply the unitary operation $I \otimes I \otimes \cdots \otimes I \otimes X$ to this quantum system. Show that this does not change the wave function of the system.

(d) Suppose that we apply a phase flip to the last of n qubits which have the wave function $|\phi\rangle$, i.e. we apply the unitary operation $I \otimes I \otimes \cdots \otimes I \otimes Z$ to this quantum system. We can express the new wave function after this operation as

$$|\phi'\rangle = \sum_{x=0, x \text{ is even}}^{2^n-1} c|x\rangle + \sum_{x=0, x \text{ is odd}}^{2^n-1} d|x\rangle$$

What are c and d?

(e) The state $|\phi'\rangle$ from part (d), can also be expressed as

$$|\phi'\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{y \cdot x} |x\rangle.$$

where y is an n bit string and $y \cdot x = y_1 x_1 + y_2 x_2 + \cdots + y_n x_n$. What is y?

(f) Suppose we apply Hadamards to all n qubits after we have applied a phase flip to the last qubit, as in parts (d) and (e). The new wave function will be $H^{\otimes n}|\phi'\rangle$, where $|\phi'\rangle$ is the wave function you calculated in parts (d) and (e). The new wave function will be a computational basis state. What state is this? (Hint: recall those nasty sums we performed in class when we where showing the orthogonality of the rows of the n qubit Hadamard.)

(g) Suppose that I secretly select an n bit string $b_1, b_2, \dots, b_n$ (where $b_i \in \{0, 1\}$.) Now you give me the n qubits with the wave function $|\phi\rangle$ (defined in the introduction to this problem.) Now I apply some phase flips to the n qubits according to the following rule: if the ith bit of my bitstring is 1, I apply a phase flip to the i qubit, otherwise I leave the ith bitstring alone. That is, I apply the unitary $Z^{b_1} \otimes Z^{b_2} \otimes \cdots \otimes Z^{b_{n-1}} \otimes Z^{b_n}$ to $|\phi\rangle$ (where recall that for a matrix M, $M^0 = I$ and $M^1 = M$.) The new wave function can be again expressed as

$$|\phi'\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{y \cdot x} |x\rangle \tag{1}$$

where y is an n bit string. What is the relationship between y and b?

(h) Suppose that I give you back the n qubits from part (g) after I have applied my phase flips as detailed in part (g). Now you apply the n qubit Hadamard to these n qubits (i.e. you apply the unitary $H^{\otimes n}$). Show that after you apply this Hadamard the n qubit wave function is a computational basis state. Indeed, show that you can learn my secret bits, $b_1, b_2, \dots, b_n$ by measuring this new state.