Quantum Computing Problem Set 2

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Problem 1: Two Qubits

In this problem we will explore two qubit systems. In particular we will become familiar with the tensor product structure of two qubits.

(a) Suppose you are given a system with a two qubit wave function $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Can this state be expressed as a separable state? That is, can you find single qubit wave functions $|\phi_1\rangle$ and $|\phi_2\rangle$ such that $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$? If yes, please find an expression for $|\phi_1\rangle$ and $|\phi_2\rangle$. If no, explain why not.

(b) Recall that the Hadamard matrix is given by

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad (1)$$

Express the two qubit unitary $H \otimes I$ in terms of the basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. That is, compute the 4 by 4 matrix elements for this matrix.

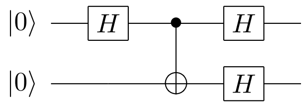
(c) Similarly express $I \otimes H$ in this same basis.

(d) Suppose that we apply $H \otimes I$ to the qubit with wave function $|\psi\rangle$ given in Problem 3, part (a) above. What is the new wave function $|\psi'\rangle = (H \otimes I)|\psi\rangle$? This will require you to multiply the 4 by 4 matrix computed above and multiply it times the state $|\psi\rangle$.

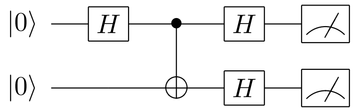
(e) Show that the $H \otimes I$ commutes with $I \otimes H$. That is, show that if you multiply these 4 by 4 matrices, then the order in which you do the multiplication does not matter: $(H \otimes I)(I \otimes H) = (I \otimes H)(H \otimes I)$.

(f) Suppose you apply $H \otimes H$ to the $|\psi\rangle$ from Problem 3, part (a). What is the resulting two qubit state? If we measure this state in the $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ basis, what are the probabilities of these four different outcomes?

(g) Finally, what two qubit wave function is the output of the follow two-qubit circuit?



If we measure these qubits in the computational basis, i.e.,



what are the probabilities of the four outcomes?

Problem 2: The Bell Basis

The Bell basis is a set of four two qubit wave functions which are of great use in quantum computation. These four basis vectors are

$$\begin{aligned} |\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \\ |\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \end{aligned} \quad (2)$$

In this problem you will become familiar with various properties of these important states.

(a) Show that the four Bell states above are orthonormal. That is show that they are all unit vectors and that the inner product of any Bell wave function with a different Bell wave function is zero.

(b) Express the four computational basis elements $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ as a combination of the four Bell states. For example, for $|00\rangle$, you should find coefficients a_0, a_1, a_2, a_3 such that $|00\rangle = a_1|\Psi_+\rangle + a_2|\Psi_-\rangle + a_3|\Phi_+\rangle + a_4|\Phi_+\rangle$.

(c) Write down the two qubit matrix U which satisfies the following conditions

$$\begin{aligned} U|00\rangle &= |\Phi_+\rangle \\ U|01\rangle &= |\Psi_+\rangle \\ U|10\rangle &= |\Phi_-\rangle \\ U|11\rangle &= |\Psi_-\rangle \end{aligned} \tag{3}$$

(d) Calculate $U^\dagger$ (i.e. what is the four by four matrix which is the conjugate transpose to U.) Use this to show that U is unitary (i.e. that $UU^\dagger = I$ where I is the four by four identity matrix.)

(e) Show that if we apply $U^\dagger$ to a generic two qubit state $|\psi\rangle$ followed by a measurement in the computational basis, then the probabilities of the four outcomes (00, 01, 10, and 11) are given by

$$Pr(00) = |\langle\Phi_+|\psi\rangle|^2, \quad Pr(01) = |\langle\Psi_+|\psi\rangle|^2, \quad Pr(10) = |\langle\Phi_-|\psi\rangle|^2, \quad \text{and} \quad Pr(11) = |\langle\Psi_-|\psi\rangle|^2 \tag{4}$$

This last result is an example of how we can perform a measurement in a *different* basis than the computational basis by using a unitary evolution followed by a measurement in the computational basis.