Quantum Computing Problem Set 1

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Problem 1: A Qubit

The basic unit of quantum information is the qubit. In this problem we will discuss a single qubit system. You are given a qubit with a wave function given by $|\psi\rangle = \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$.
(a) What is the bra $\langle\psi|$ corresponding to this state?
(b) Show that this state is a normalized wave function, that is show $\langle\psi|\psi\rangle = 1$.
(c) If you measure this qubit in the computational basis $|0\rangle$, $|1\rangle$, what are the probabilities of obtaining each of the two outcomes for this measurement?
(d) Define the two basis states $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$. Calculate the inner product of these states with $|\psi\rangle$. That is, find $\langle+|\psi\rangle$ and $\langle-|\psi\rangle$.
(e) Verify that $|+\rangle$ and $|-\rangle$ are orthogonal: $\langle+|-\rangle = 0$. Verify that $|+\rangle$ and $|-\rangle$ are also both properly normalized. In other words show that $\langle+|+\rangle = \langle-|-\rangle = 1$.
(f) Suppose we express $|\psi\rangle$ in the $|+\rangle$, $|-\rangle$ basis as $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$. Show that $\alpha = \langle+|\psi\rangle$ and $\beta = \langle-|\psi\rangle$. Hint: form the bracket $\langle+|\psi\rangle$ and use the fact that $|+\rangle$ and $|-\rangle$ are orthogonal.
(g) Use the previous three parts to express $|\psi\rangle$ as $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$ but with complex numbers instead of α and β.
(h) If we measure the state $|\psi\rangle$ in the $|+\rangle$ and $|-\rangle$ basis, what are the probabilities of obtaining each of these two outcomes?

Problem 2: Single Qubit Matrices

The basic manipulations of a qubit will be two by two matrices. In this problem we will become familiar a particular single qubit matrix.
(a) Let

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Calculate the complex conjugate to U: U^*. Your answer will be a two by two matrix.
(b) Calculate the adjoint to U: $U^\dagger$. Again your answer will be a two by two matrix.
(c) Verify that U is unitary. That is, show that $UU^\dagger = I$. This means that U is a valid quantum evolution.
(d) If we apply the evolution corresponding to U to $|\psi\rangle$, our new state is $|\psi'\rangle = U|\psi\rangle$. What is $|\psi'\rangle$ if $|\psi\rangle = \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$?
(e) Calculate U^2 and U^4. One of these should be negative the identity matrix.
(f) Suppose we apply the evolution corresponding to U twice to a qubit with the wave function $|\psi\rangle = \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$. What is the state after this application?
(g) Recall that the Pauli matrices are the following two by two matrices:

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \text{ and, } \sigma_3 = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1)$$

Express U as a linear combination of these matrices. That is, express U as

$$U = s_0\sigma_0 + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 \quad (2)$$

for specific, complex s_i's.
(h) Calculate $UZU^\dagger$. Your answer will be a two by two matrix. In particular it will be equal to one of the Pauli matrices.