

Quantum Computing Problem Set 3, Problem 3 Solution

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Problem 3: Teleporting an Entangled State

In this problem we will run through the problem of teleporting one half of an entangled quantum state. Alice will be teleporting to Bob. Suppose that Alice has two qubits whose wave function is given by

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle \quad (1)$$

and that Alice and Bob share the two qubit entangled state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (2)$$

To be clear, Alice has three qubits and Bob has one. Recall that the Bell basis is given by the four basis states

$$\begin{aligned} |\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), & |\Psi_-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \\ |\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), & |\Phi_-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \end{aligned} \quad (3)$$

If we label the qubits one through four, then Alice has the first three qubits and Bob has the final qubit and the initial state is $|\psi\rangle|\Phi_+\rangle$.

(a) Suppose that Alice measures her second and third qubits in the Bell basis. What are the probabilities of these four outcomes?

Solution: The state of the four qubits is

$$|v\rangle = (\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The measurement operators for the four outcomes are

$$\begin{aligned} M_0 &= I \otimes |\Phi_+\rangle\langle\Phi_+| \otimes I \\ M_1 &= I \otimes |\Phi_-\rangle\langle\Phi_-| \otimes I \\ M_2 &= I \otimes |\Psi_+\rangle\langle\Psi_+| \otimes I \\ M_3 &= I \otimes |\Psi_-\rangle\langle\Psi_-| \otimes I \end{aligned}$$

The probability associated with each outcome i is therefore $\langle v|M_i|v\rangle$. We will first calculate $I \otimes M_i \otimes I|v\rangle$ on each of these states.

$$\begin{aligned} M_0|v\rangle &= (I \otimes |\Phi_+\rangle\langle\Phi_+| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ M_1|v\rangle &= (I \otimes |\Phi_-\rangle\langle\Phi_-| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ M_2|v\rangle &= (I \otimes |\Psi_+\rangle\langle\Psi_+| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ M_3|v\rangle &= (I \otimes |\Psi_-\rangle\langle\Psi_-| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \end{aligned}$$

Let's look at the first of these

$$(I \otimes |\Phi_+\rangle\langle\Phi_+| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The important term here is the $\langle\Phi_+|$ term. It will only have nonzero inner product when the second and third qubits are in the same computational basis state. In other words this will only have overlap with the state for the terms $\frac{\alpha}{\sqrt{2}}|0000\rangle$ and $\frac{\beta}{\sqrt{2}}|1111\rangle$. Thus we find that

$$(I \otimes |\Phi_+\rangle\langle\Phi_+| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{\alpha}{2}|0\rangle \otimes |\Phi_+\rangle \otimes |0\rangle + \frac{\beta}{2}|1\rangle \otimes |\Phi_+\rangle \otimes |1\rangle$$

Next we need to take the inner product of this state with $\langle v|$. By expanding out $\langle v|$ and the above state we find

$$\langle v|M_0|v\rangle = \frac{\alpha^*\alpha}{4} + \frac{\beta^*\beta}{4} = \frac{1}{4}$$

Similarly we can calculate the other $M_i|v\rangle$'s. These are

$$\begin{aligned} (I \otimes |\Phi_-\rangle\langle\Phi_-| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) &= \frac{\alpha}{2}|0\rangle \otimes |\Phi_-\rangle \otimes |0\rangle - \frac{\beta}{2}|1\rangle \otimes |\Phi_-\rangle \otimes |1\rangle \\ (I \otimes |\Psi_+\rangle\langle\Psi_+| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) &= \frac{\alpha}{2}|0\rangle \otimes |\Psi_+\rangle \otimes |1\rangle + \frac{\beta}{2}|1\rangle \otimes |\Psi_+\rangle \otimes |0\rangle \\ (I \otimes |\Psi_-\rangle\langle\Psi_-| \otimes I)(\alpha|00\rangle + \beta|11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) &= \frac{\alpha}{2}|0\rangle \otimes |\Psi_-\rangle \otimes |1\rangle - \frac{\beta}{2}|1\rangle \otimes |\Psi_-\rangle \otimes |0\rangle \end{aligned}$$

Taking the inner products with $\langle v|$, yields

$$\begin{aligned} \langle v|M_1|v\rangle &= \frac{\alpha^*\alpha}{4} + \frac{\beta^*\beta}{4} = \frac{1}{4} \\ \langle v|M_2|v\rangle &= \frac{\alpha^*\alpha}{4} + \frac{\beta^*\beta}{4} = \frac{1}{4} \\ \langle v|M_3|v\rangle &= \frac{\alpha^*\alpha}{4} + \frac{\beta^*\beta}{4} = \frac{1}{4} \end{aligned}$$

Thus the probability of all four outcomes is $\frac{1}{4}$.

- (b) What is the full quantum wave function for all four qubits after Alice measures in the Bell basis for each of these four outcomes?

Solution: The wave function for each of the four qubits is given by

$$|v_i\rangle = \frac{(I \otimes M_i \otimes I)|v\rangle}{\sqrt{Pr(i)}}$$

In our case, $\sqrt{Pr(i)}$ is $\frac{1}{2}$. We have calculated these states in the previous part. We just need to divide by this probability. The states are thus

$$\begin{aligned} |v_0\rangle &= \alpha|0\rangle \otimes |\Phi_+\rangle \otimes |0\rangle + \beta|1\rangle \otimes |\Phi_+\rangle \otimes |1\rangle \\ |v_1\rangle &= \alpha|0\rangle \otimes |\Phi_-\rangle \otimes |0\rangle - \beta|1\rangle \otimes |\Phi_-\rangle \otimes |1\rangle \\ |v_2\rangle &= \alpha|0\rangle \otimes |\Psi_+\rangle \otimes |1\rangle + \beta|1\rangle \otimes |\Psi_+\rangle \otimes |0\rangle \\ |v_3\rangle &= \alpha|0\rangle \otimes |\Psi_-\rangle \otimes |1\rangle - \beta|1\rangle \otimes |\Psi_-\rangle \otimes |0\rangle \end{aligned}$$

- (c) Show that if Alice gets the measurement outcome corresponding to $|\Phi_+\rangle$, then the state of the first qubit and the fourth qubit is $|\psi\rangle$.

Solution: The wave function in this case is $|v_0\rangle$ from the previous problem,

$$|v_0\rangle = \alpha|0\rangle \otimes |\Phi_+\rangle \otimes |0\rangle + \beta|1\rangle \otimes |\Phi_+\rangle \otimes |1\rangle$$

Label each of the qubits that the state is acting on as 1, 2, 3 or 4.

$$|v_0\rangle = \alpha|0\rangle_1 \otimes |\Phi_+\rangle_{23} \otimes |0\rangle_4 + \beta|1\rangle_1 \otimes |\Phi_+\rangle_{23} \otimes |1\rangle_4$$

Which we can reexpress as

$$|v_0\rangle = \alpha|00\rangle_{14} \otimes |\Phi_+\rangle_{23} + \beta|11\rangle_{14} \otimes |\Phi_+\rangle_{23}$$

which is separable across the 14, 23 divide:

$$|v_0\rangle = (\alpha|00\rangle_{14} + \beta|11\rangle_{14}) \otimes |\Phi_+\rangle_{23}$$

Which, as claimed, is the state $|\psi\rangle$ for qubits 1 and 4.

- (d) Show that if Alice gets the measurement outcome corresponding to $|\Psi_+\rangle$, then the state of the first qubit and the fourth qubit is $(I \otimes X)|\psi\rangle$.

Solution: This calculation is similar to the previous problem. If we carry out the same procedure of relabeling for $|v_2\rangle$, we end up with

$$|v_2\rangle = (\alpha|01\rangle_{14} + \beta|10\rangle_{14}) \otimes |\Psi_+\rangle_{23}$$

But notice that $(I \otimes X)|\psi\rangle = \alpha|01\rangle + \beta|10\rangle$. But this is just the state we have in our expression for $|v_2\rangle$.

- (e) Describe a procedure for teleporting one half of Alice's entangled state $|\psi\rangle$. That is, show how Alice can send two classical bits of communication to Bob and Bob can apply an appropriate unitary operation, such that at the end of the procedure, the first and fourth qubits are in the state $|\psi\rangle$. (Notice that we have half solved this problem using the results in part (c) and part (d).)

Solution: We see that if Alice carries out the Bell basis measurement described in the problem (on the second and third qubits), then already we sometimes teleport the state. In particular if the outcome of the Bell basis measurement is $|\Phi_+\rangle$, then the state is, as desired, the original entangled state, but now shared between the 1 and 4 qubit. If the outcome was $|\Psi_+\rangle$, then the state was $|\psi\rangle$ except for the fact that we needed to apply X to Bob's qubit. Similarly we can calculate that if the outcome was $|\Phi_-\rangle$, then the first and fourth qubits have the wave function $(I \otimes Z)(I \otimes X)|\psi\rangle$. If the outcome was $|\Psi_-\rangle$, then the first and fourth qubits have the wave function $(I \otimes Z)(I \otimes X)|\psi\rangle$. Thus we can perform teleportation just as we did in standard qubit. In particular, Alice sends two bits to Bob corresponding to her Bell basis outcome. Depending on what Bob gets, he then applies the appropriate unitary to his qubit to produce $|\psi\rangle$ stored between the first and fourth qubit. His rules are: outcome $|\Phi_+\rangle$: do nothing, outcome $|\Phi_-\rangle$, apply a Z , outcome $|\Psi_+\rangle$: apply X , and outcome $|\Psi_-\rangle$, apply a ZX .