CSEP 590tv In Class Problems, July 20, 2005

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1. Alice has the first qubit of the two qubit system with wave function $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Now suppose that Alice applies X to her qubit. What is the new two qubit wave function? Is this state orthogonal to $|\Phi_+\rangle$?

Solution: Alice is applying the unitary operation $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to her qubit (the first one.) Thus the unitary operation being applied is

$$U = X \otimes I$$

where I is the two dimensional identity matrix.

So we want to calculate

$$(X \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

which we can evaluate as

$$(X \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}((X \otimes I)|00\rangle + (X \otimes I)|11\rangle$$

Now $X \otimes I|00\rangle = |10\rangle$ and $(X \otimes I)|11\rangle = |01\rangle$. Thus we find that

$$|v\rangle = (X \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

Which is our answer.

Now lets check whether this state is orthogonal to $|\Phi_+\rangle$. We want to calculate

$$\langle \Phi_+|v\rangle$$

As a first step, we find the bra:

$$\langle \Phi_+| = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)$$

And so

$$\langle \Phi_+|v\rangle = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \frac{1}{2}(\langle 00|10\rangle + \langle 00|01\rangle + \langle 11|10\rangle + \langle 11|01\rangle) = 0$$

because all those inner products are orthogonal (the computational basis states are orthogonal.) Thus we indeed find that the new state $|v\rangle$ is orthogonal to $|\Phi_+\rangle$.

What if Alice had applied Z to her qubit, what is the new two qubit wave function? Is this state orthogonal to $|\Phi_+\rangle$?

Solution: Recall that $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We proceed as above

$$(Z \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}((Z \otimes I)|00\rangle + (Z \otimes I)|11\rangle)$$

But now $(Z \otimes I)|00\rangle = |00\rangle$ and $(Z \otimes I)|11\rangle = -|11\rangle$. Substituting we find

$$|w\rangle = (Z \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Now is this state orthogonal to $|Phi_+\rangle$? Calculate the inner product:

$$\langle w|\Phi_+\rangle = \frac{1}{\sqrt{2}}(\langle 00| - \langle 11|)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

which we expand as

$$\langle w|\Phi_+\rangle = \frac{1}{2}(\langle 00|00\rangle + \langle 00|11\rangle - \langle 11|00\rangle - \langle 11|11\rangle$$

Recall that $\langle 00|00\rangle = \langle 11|11\rangle = 1$ and $\langle 00|11\rangle = \langle 11|00\rangle = 0$. Thus

$$\langle w|\Phi_+\rangle = \frac{1}{2}(1 + 0 - 0 - 1) = 0$$

So again they are orthogonal!

2. Suppose we have n qubits with the wave function $|\phi_x\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x}|y\rangle$. What is the new n qubit wave function if we apply the n qubit unitary $Z^{\otimes n}$ to this state?

Solution: We are starting with the state

$$|\phi_x\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x}|y\rangle$$

and we want to calculate

$$(Z^{\otimes n})|\phi_x\rangle$$

How to do this? First, note that

$$(Z^{\otimes n})|\phi_x\rangle = (Z^{\otimes n})\frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x}|y\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x}(Z^{\otimes n})|y\rangle$$

So we need to figure out how to calculate $(Z^{\otimes N})|y\rangle$. Recall that $|y\rangle$ is shorthand for the bitstring $|y_1, y_2, \ldots, y_n\rangle$. So this allows us to write

$$(Z^{\otimes n})|y\rangle = (Z^{\otimes n})|y_1, y_2, \ldots, y_n\rangle = (Z^{\otimes n})|y_1\rangle \otimes |y_2\rangle \otimes \cdots \otimes |y_n\rangle$$

Now we can act tensor wise with the Z operator

$$(Z^{\otimes n})|y\rangle = (Z|y_1\rangle) \otimes (Z|y_2\rangle) \otimes \cdots \otimes (Z|y_n\rangle)$$

What are these things $Z|y_i\rangle$? Well if $y_i = 0$, then $Z|y_i = 0\rangle = |y_i\rangle$, but if $y_i = 1$, then $Z|y_i = 1\rangle = -|y_i = 1\rangle$. So we can express $Z|y_i\rangle = (-1)^{y_i}|y_i\rangle$ (cus $(-1)^0 = 1$ and $(-1)^1 = -1$.) OK continuing, we can now substitute these new expressions:

$$(Z^{\otimes n})|y\rangle = ((-1)^{y_1}|y_1\rangle) \otimes ((-1)^{y_2}|y_2\rangle) \otimes \cdots \otimes ((-1)^{y_n}|y_n\rangle)$$

We can pull all these (-1) expressions out front:

$$(Z^{\otimes n})|y\rangle = (-1)^{y_1}(-1)^{y_2}\cdots(-1)^{y_n}|y_1\rangle \otimes |y_2\rangle \otimes \cdots \otimes |y_n\rangle$$

And then using the fact that $(-1)^a(-1)^b = (-1)^{a+b}$, we find that

$$(Z^{\otimes n})|y\rangle = (-1)^{y_1+y_2+\cdots+y_n}|y_1\rangle \otimes |y_2\rangle \otimes \cdots \otimes |y_n\rangle$$

Now define the all 1s bitstring $\bar{1} = 111\ldots1$. Then we can express that term in the exponent as $y_1+y_2+\cdots+y_n = \bar{1}\cdot y$. Collecting the tensor product of the bits together, we have found

$$(Z^{\otimes n})|y\rangle = (-1)^{y\cdot\bar{1}}|y\rangle$$

Cool.

Now we just need to substitute this back into our big expression:

$$(Z^{\otimes n})|\phi_x\rangle = (Z^{\otimes n})\frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x}|y\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x}(Z^{\otimes n})|y\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x}(-1)^{y\cdot\bar{1}}|y\rangle$$

Again using $(-1)^a(-1)^b = (-1)^{a+b}$, we find that

$$(Z^{\otimes n})|\phi_x\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}(-1)^{y\cdot x+y\cdot\bar{1}}|y\rangle$$