Quantum Computing
Problem Set 5 Solutions

1. (a) Many ways to check this but is should be obvious that the answer is $s_1 = 1, s_2 = 1$.

(b)

$$|v\rangle = \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |110\rangle)$$

and discarding all the 0 terms

$$\begin{aligned} |v'\rangle = U|v\rangle &= \frac{1}{2}(U|000\rangle + U|010\rangle + U|100\rangle + U|110\rangle) \\ &= \frac{1}{2}(|0,0,0 \oplus f(0,0)\rangle + |0,1,0 \oplus f(0,1)\rangle + |1,0,0 \oplus f(1,0)\rangle + |1,1,0 \oplus f(1,1)\rangle) \\ &= \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \end{aligned}$$

(c)

$$\begin{aligned} P_0 &= I \otimes I \otimes |0\rangle\langle 0| \\ &= |000\rangle\langle 000| + |010\rangle\langle 010| + |100\rangle\langle 100| + |110\rangle\langle 110| \\ P_0|v'\rangle &= \frac{1}{2}(|000\rangle + |110\rangle) \\ Pr[0] &= \langle v'|\frac{1}{2}(|000\rangle + |110\rangle) \\ &= \frac{1}{2}(\langle 000| + \langle 011| + \langle 101| + \langle 110|)\frac{1}{2}(|000\rangle + |110\rangle) \\ &= \frac{1}{4}(1+1) = \frac{1}{2} \end{aligned}$$

When we get outcome 0 , the new wave function will be

$$\frac{\frac{1}{2}(|000\rangle + |110\rangle)}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |0\rangle$$

Therefore the state of the first two qubits is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

(d)

$$H \otimes H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H \otimes H(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

If we now measure in the computational basis, we get

$$\begin{aligned}
Pr[00] = && ||\langle 00|\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)||^2 = \frac{1}{2} \\
Pr[01] = && 0 \\
Pr[10] = && 0 \\
Pr[11] = && 1
\end{aligned}$$

The two possible outcomes are 00 and 11, with probability 1/2.

(e) Since $s_1 + s_2 = 0 \ mod 2$, the only non-zero solution is $s_1 = 1, s_2 = 1$, the same as in part a.

2. (a)

$$\begin{aligned}
f_{0,0}(x_1, x_2) = && (1 - x_1)(1 - x_2) \\
f_{0,1}(x_1, x_2) = && (1 - x_1)x_2 \\
f_{1,0}(x_1, x_2) = && x_1(1 - x_2) \\
f_{1,1}(x_1, x_2) = && x_1 x_2
\end{aligned}$$

(b) By discarding the 0 terms we get

$$\begin{aligned}
U_{1,0}|011\rangle &= |0, 1, 1 \oplus f_{1,0}(0, 1)\rangle \\
&= |0, 1, 1 \oplus 0\rangle \\
&= |011\rangle
\end{aligned}$$

(c)

$$\begin{aligned}
|v'\rangle &= U_{\alpha_1,\alpha_2}|v\rangle \\
&= U_{\alpha_1,\alpha_2}\frac{1}{2\sqrt{2}}\sum_{x_1=0}^{1}\sum_{x_2=0}^{1}(|x_1, x_2, 0\rangle - |x_1, x_2, 1\rangle) \\
&= \frac{1}{2}\sum_{x_1=0}^{1}\sum_{x_2=0}^{1}U_{\alpha_1,\alpha_2}(|x_1, x_2\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle))
\end{aligned}$$

If $x_1, x_2 = \alpha_1, \alpha_2$, then $f_{\alpha_1,\alpha_2}(x_1, x_2) = 1$. Effectively this will flip the third qubit, so the inner term will be

$$|x_1, x_2\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$$

In the rest of the cases, the inner term remains

$$|x_1, x_2\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

We can combine these cases into

$$(-1)^{f_{\alpha_1,\alpha_2}(x_1,x_2)}|x_1,x_2\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

and the result follows.

You can also get this more directly by applying the phase kickback argument.

(d) Suppose we have two possible pairs of (a_1, a_2); call these (a_1, a_2) and (a_1', a_2'). Then we form the inner product:

$$\begin{aligned}
\langle a_1', a_2' | a_1, a_2\rangle &= \left[\frac{1}{2}\sum_{x_1',x_2'=0}^{1}(-1)^{f_{a_1,a_2}(x_1',x_2')}\langle x_1', x_2'|\right]\left[\frac{1}{2}\sum_{x_1,x_2=0}^{1}(-1)^{f_{a_1,a_2}(x_1,x_2)}|x_1,x_2\rangle\right] \\
&= \frac{1}{4}\sum_{x_1',x_2'=0}^{1}\sum_{x_1,x_2=0}^{1}(-1)^{f_{a_1,a_2}(x_1',x_2')+f_{a_1,a_2}(x_1,x_2)}\langle x_1', x_2'|x_1, x_2\rangle \\
&= \frac{1}{4}\sum_{x_1',x_2'=0}^{1}\sum_{x_1,x_2=0}^{1}(-1)^{f_{a_1,a_2}(x_1',x_2')+f_{a_1,a_2}(x_1,x_2)}\delta_{(x_1',x_2'),(x_1,x_2)} \\
&= \frac{1}{4}\sum_{x_1',x_2'=0}^{1}(-1)^{2f_{a_1,a_2}(x_1,x_2)}\delta_{x_1',x_2',x_1,x_2} \\
&= \delta_{x_1',x_2',x_1,x_2}
\end{aligned}$$

Thus we have 0 when $x' \neq x$ and 1 when $x' = x$. So we have an orthonormal basis.

(e) There are many ways to derive this unitary. For example you can consider the unitary elements as unknowns and solve the equations or work backwards and eyeball the unitary that satisfy the requirements.

You should end up with

$$U = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Let's verify the four different cases

i. If $(\alpha_1, \alpha_2) = (0, 0)$, then $|\alpha_1, \alpha_2\rangle = \frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.

$$U|\alpha_1, \alpha_2\rangle = |00\rangle$$

ii. If $(\alpha_1, \alpha_2) = (0, 1)$, then $|\alpha_1, \alpha_2\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$.

$$U|\alpha_1, \alpha_2\rangle = |01\rangle$$

iii. If $(\alpha_1, \alpha_2) = (1, 0)$, then $|\alpha_1, \alpha_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$.

$$U|\alpha_1, \alpha_2\rangle = |10\rangle$$

iv. If $(\alpha_1, \alpha_2) = (1, 1)$, then $|\alpha_1, \alpha_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$.

$$U|\alpha_1, \alpha_2\rangle = |11\rangle$$