Quantum Computing
Problem Set 4 Solutions

1. (a)

$$\begin{aligned}
X^{\otimes n}|\phi\rangle &= X^{\otimes n}\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}X^{\otimes n}|x\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(X|x_1\rangle\otimes X|x_2\rangle\otimes\cdots\otimes X|x_n\rangle) \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(|\overline{x_1}\rangle\otimes|\overline{x_2}\rangle\otimes\cdots\otimes|\overline{x_n}\rangle) \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|\overline{x}\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle = |\phi\rangle
\end{aligned}$$

where the last step follows from the fact that the sum of all states is equal to the sum of all states in reverse order, which is what we get by flipping all qubits.

(b)

$$
\begin{aligned}
Z^{\otimes n}|\phi\rangle &= Z^{\otimes n}\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}Z^{\otimes n}|x\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\left(Z|x_1\rangle\otimes Z|x_2\rangle\otimes\cdots\otimes Z|x_n\rangle\right) \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\left((-1)^{x_1}|x_1\rangle\otimes(-1)^{x_2}|x_2\rangle\otimes\cdots\otimes(-1)^{x_n}|x_n\rangle\right) \\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(-1)^{x_1}(-1)^{x_2}\cdots(-1)^{x_n}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|x_n\rangle\right) \\
&= \sum_{x=0}^{2^n-1}\frac{1}{\sqrt{2^n}}(-1)^{x_1+x_2+\cdots+x_n}|x\rangle
\end{aligned}
$$

so $a_x = \frac{1}{\sqrt{2^n}}(-1)^{x_1+x_2+\cdots+x_n}$.

(c)

$$
(I\otimes I\otimes\cdots\otimes X)\,|\phi\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes X|x_n\rangle\right)
$$

$$
=\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|\overline{x_n}\rangle\right)
$$

$$
=\frac{1}{\sqrt{2^n}}\sum_{x=0,x\text{ is even}}^{2^n-1}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|\overline{x_n}\rangle\right)+\frac{1}{\sqrt{2^n}}\sum_{x=0,x\text{ is odd}}^{2^n-1}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|\overline{x_n}\rangle\right)
$$

$$
=\frac{1}{\sqrt{2^n}}\sum_{x=0,x\text{ is odd}}^{2^n-1}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|x_n\rangle\right)+\frac{1}{\sqrt{2^n}}\sum_{x=0,x\text{ is even}}^{2^n-1}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|x_n\rangle\right)
$$

$$
=\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\left(|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|x_n\rangle\right)=|\phi\rangle
$$

(d)

$$(I \otimes I \otimes \cdots \otimes Z)\,|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes Z|x_n\rangle)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes (-1)^{x_n}|x_n\rangle)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is even}}^{2^n-1} (|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes (-1)^{x_n}|x_n\rangle) + \frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is odd}}^{2^n-1} (|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes (-1)^{x_n}|x_n\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is even}}^{2^n-1} (|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle) + \frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is odd}}^{2^n-1} (|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes (-1)|x_n\rangle)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is even}}^{2^n-1} |x\rangle + \frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is odd}}^{2^n-1} -|x\rangle$$

so $c = \frac{1}{\sqrt{2^n}}, d = c = \frac{-1}{\sqrt{2^n}}$.

(e)

$$\frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is even}}^{2^n-1} |x\rangle + \frac{1}{\sqrt{2^n}} \sum_{x=0, x \text{ is odd}}^{2^n-1} -|x\rangle =$$
$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{x_n}|x\rangle$$

so we want $y \cdot x = x_n$ which means that y must be equal to $00\ldots01$.

(f)

$$\begin{aligned}
H^{\otimes n}|\phi'\rangle &= Z^{\otimes n}\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{y\cdot x} H^{\otimes n}|x\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{y\cdot x} \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^n-1} (-1)^{z\cdot x}|z\rangle \\
&= \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^n-1} (-1)^{(z\oplus 1)\cdot x}|z\rangle \\
&= \frac{1}{2^n} \sum_{z=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{(z\oplus 1)\cdot x}|z\rangle
\end{aligned}$$

as seen in class

$$\sum_{x=0}^{2^n-1} (-1)^{(z\oplus 1)\cdot x} = \delta_{(z\oplus 1),0}$$

so we finally get

$$H^{\otimes n}|\phi'\rangle = \frac{1}{2^n}\sum_{z=0}^{2^n-1}\delta_{(z\oplus 1),0}|z\rangle = |1\rangle$$

(g)

$$\begin{aligned}
Z^{b_1}\otimes Z^{b_2}\otimes\cdots\otimes Z^{b_n}|\phi\rangle &= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\Big(Z^{b_1}|x_1\rangle\otimes Z^{b_2}|x_2\rangle\otimes\cdots\otimes Z^{b_n}|x_n\rangle\Big)\\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\Big((-1)^{b_1x_1}|x_1\rangle\otimes(-1)^{b_2x_2}|x_2\rangle\otimes\cdots\otimes(-1)^{b_nx_n}|x_n\rangle\Big)\\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\Big((-1)^{b_1x_1+b_2x_2+\cdots+b_nx_n}|x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|x_n\rangle\Big)\\
&= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}\Big((-1)^{b\cdot x}|x\rangle\Big)
\end{aligned}$$

so $y = b$.

(h)

$$\begin{aligned}
H^{\otimes n}|\phi'\rangle &= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}(-1)^{b\cdot x}H^{\otimes n}|x\rangle\\
&= \frac{1}{\sqrt{2^n}}\sum_{z=0}^{2^n-1}\sum_{x=0}^{2^n-1}(-1)^{(z\oplus b)\cdot x}|z\rangle\\
&= \frac{1}{\sqrt{2^n}}\sum_{z=0}^{2^n-1}\delta_{(z\oplus b),0}|z\rangle\\
&= |b\rangle
\end{aligned}$$

which means that if we measure in the computational basis we will be able to retrieve b by the measurement.