Quantum Computing
Problem Set 3 Solutions

1. (a) We start by figuring out the two projectors.

$$P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|$$

$$\begin{aligned} P_0 \otimes I \otimes I &= |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= |000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011| \end{aligned}$$

$$\begin{aligned} P_0 \otimes I \otimes I &= |1\rangle\langle 1| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= |100\rangle\langle 100| + |101\rangle\langle 101| + |110\rangle\langle 110| + |111\rangle\langle 111| \end{aligned}$$

$$\begin{aligned} (P_0 \otimes I \otimes I)|GHZ\rangle &= (|000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011|)\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\ &= \frac{1}{\sqrt{2}}|000\rangle \\ (P_1 \otimes I \otimes I)|GHZ\rangle &= (|100\rangle\langle 100| + |101\rangle\langle 101| + |110\rangle\langle 110| + |111\rangle\langle 111|)\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\ &= \frac{1}{\sqrt{2}}|111\rangle \end{aligned}$$

$$\begin{aligned} Pr[0] &= \langle GHZ|(\frac{1}{\sqrt{2}}|000\rangle) = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)(\frac{1}{\sqrt{2}}|000\rangle) = \frac{1}{2} \\ Pr[1] &= \langle GHZ|(\frac{1}{\sqrt{2}}|111\rangle) = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)(\frac{1}{\sqrt{2}}|111\rangle) = \frac{1}{2} \end{aligned}$$

If the outcome is 0 the resulting wave function is

$$|GHZ'\rangle = \frac{\frac{1}{\sqrt{2}}|000\rangle}{\frac{1}{\sqrt{2}}} = |000\rangle$$

and if the outcome is 1

$$|GHZ'\rangle = \frac{\frac{1}{\sqrt{2}}|111\rangle}{\frac{1}{\sqrt{2}}} = |111\rangle$$

(b)

$$\begin{aligned}
P_+ &= |+\rangle\langle+| = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) \\
&= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \\
P_- &= |-\rangle\langle-| = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|) \\
&= \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)
\end{aligned}$$

$$\begin{aligned}
P_+ \otimes I \otimes I = &\frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&\frac{1}{2}(|000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011| + \\
&|000\rangle\langle 100| + |001\rangle\langle 101| + |010\rangle\langle 110| + |011\rangle\langle 111| + \\
&|100\rangle\langle 000| + |101\rangle\langle 001| + |110\rangle\langle 010| + |111\rangle\langle 011| + \\
&|100\rangle\langle 100| + |101\rangle\langle 101| + |110\rangle\langle 110| + |111\rangle\langle 111| \\
P_- \otimes I \otimes I = &\frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&= \frac{1}{2}(|000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011| + \\
&-|000\rangle\langle 100| - |001\rangle\langle 101| - |010\rangle\langle 110| - |011\rangle\langle 111| \\
&-|100\rangle\langle 000| - |101\rangle\langle 001| - |110\rangle\langle 010| - |111\rangle\langle 011| + \\
&|100\rangle\langle 100| + |101\rangle\langle 101| + |110\rangle\langle 110| + |111\rangle\langle 111|
\end{aligned}$$

$$\begin{aligned}
(P_+ \otimes I \otimes I)|GHZ\rangle &= \frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle + |011\rangle + |111\rangle) \\
(P_- \otimes I \otimes I)|GHZ\rangle &= \frac{1}{2\sqrt{2}}(|000\rangle - |100\rangle - |011\rangle + |111\rangle)
\end{aligned}$$

$$\begin{aligned}
Pr[+] &= \langle GHZ|\frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle + |011\rangle + |111\rangle) = \frac{1}{2} \\
Pr[-] &= \langle GHZ|\frac{1}{2\sqrt{2}}(|000\rangle - |100\rangle - |011\rangle + |111\rangle) = \frac{1}{2}
\end{aligned}$$

If the outcome is + the resulting wave function is

$$|GHZ'\rangle = \frac{1}{2}(|000\rangle + |100\rangle + |011\rangle + |111\rangle)$$

and if the outcome is -

$$|GHZ'\rangle = \frac{1}{2}(|000\rangle - |100\rangle - |011\rangle + |111\rangle)$$

(c) We will be calculating the tensor product of 3 projectors. The general form will be $P_a \otimes P_b \otimes P_c$. In this product, most of the terms will be eventually canceled in the inner product with $|GHZ\rangle$. The only interesting terms will be $|000\rangle\langle 000|$, $|000\rangle\langle 111|$, $|111\rangle\langle 000|$ and $|111\rangle\langle 111|$. We will get

$$P_a \otimes P_b \otimes P_c = \frac{1}{8}(|000\rangle\langle 000| + abc|000\rangle\langle 111| + abc|111\rangle\langle 000| + |111\rangle\langle 111| + \ldots)$$

where $a, b, c = \pm 1$.

$$\begin{aligned} \langle GHZ|P_a \otimes P_b \otimes P_c|GHZ\rangle &= \langle GHZ|\frac{1}{8\sqrt{2}}(|000\rangle + abc|000\rangle + abc|111\rangle + |111\rangle) \\ &= \frac{1}{8}(1 + abc) \end{aligned}$$

The resulting probabilities are 1/4 or 0 depending on a, b, c. We get the following results

$$\begin{aligned} Pr[+ + +] &= 1/4 \\ Pr[+ + -] &= 0 \\ Pr[+ - +] &= 0 \\ Pr[- + +] &= 0 \\ Pr[+ - -] &= 1/4 \\ Pr[- + -] &= 1/4 \\ Pr[- - +] &= 1/4 \\ Pr[- - -] &= 0 \end{aligned}$$

(d) We start applying a Hadamard gate on the first qubit. We get

$$|000\rangle \rightarrow \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle)$$

Now, we apply a NOT on the second qubit, controlled by the first.

$$\frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \rightarrow \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)$$

and finally we apply a NOT on the third cubit, controlled by the second.

$$\frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) \rightarrow \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

We could work out each unitary in the formal way, but knowing what each unitary does, we can easily get the same result.

2. We will verify that the circuits are equivalent by calculating the resulting unitary. Notice that in the matrix notation, the first gate comes as the second matrix.

(a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{X\otimes I} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{X\otimes X} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{I\otimes X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{I\otimes X} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{Z\otimes I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{Z\otimes I} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{I\otimes Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{I\otimes Z} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$