# Quantum Computing
# Problem Set 2 Solutions

**1.** (a) Suppose the two qubits are $|\phi_1\rangle = a_1|0\rangle + b_1|1\rangle$ and $|\phi_2\rangle = a_2|0\rangle + b_2|1\rangle$.

$$|\phi_1\rangle \otimes |\phi_2\rangle = a_1a_2|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + b_1b_2|11\rangle$$

We need a solution such that $a_1a_2 = a_1b_2 = a_2b_1 = b_1b_2 = \frac{1}{2}$. Indeed, for $a_1 = a_2 = b_1 = b_2 = \frac{1}{\sqrt{2}}$ we can express $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$ where

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\ ,\ |\phi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

(b)

$$H \otimes I = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

(c)

$$I \otimes H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

(d)

$$|\psi'\rangle = (H \otimes I)|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$$

(e)

$$(H \otimes I)(I \otimes H) = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

$$(I \otimes H)(H \otimes I) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

(f)

$$H \otimes H = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

$$(H \otimes H)|\psi\rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly the probabilities are $Pr[00] = 1, Pr[01] = Pr[10] = Pr[11] = 0$.
First, we apply $H \otimes I$ to $|00\rangle$. We get

$$(H \otimes I)|00\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

After the controlled NOT we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Finally, after with apply $H \otimes H$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

The probabilities of each outcome are $Pr[00] = Pr[11] = 1/2, Pr[01] = Pr[10] = 1/2$.

**2.** (a) We verify that the Bell states are orthonormal

$$\begin{aligned}
\langle \Psi_+ | \Psi_+ \rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 \\
\langle \Psi_- | \Psi_- \rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + (\frac{-1}{\sqrt{2}})(\frac{-1}{\sqrt{2}}) = 1 \\
\langle \Phi_+ | \Phi_+ \rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 \\
\langle \Phi_- | \Phi_- \rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + (\frac{-1}{\sqrt{2}})(\frac{-1}{\sqrt{2}}) = 1
\end{aligned}$$

Observe that when there are no imaginary components, $\langle x|y\rangle = \langle y|x\rangle$.

$$\begin{aligned}
\langle \Psi_+ | \Psi_- \rangle = \langle \Psi_- | \Psi_+ \rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(\frac{-1}{\sqrt{2}}) = 0 \\
\langle \Psi_+ | \Phi_+ \rangle = \langle \Phi_+ | \Psi_+ \rangle &= 0 + 0 + 0 + 0 = 0 \\
\langle \Psi_+ | \Phi_- \rangle = \langle \Phi_- | \Psi_+ \rangle &= 0 + 0 + 0 + 0 = 0 \\
\langle \Psi_- | \Phi_+ \rangle = \langle \Phi_+ | \Psi_- \rangle &= 0 + 0 + 0 + 0 = 0 \\
\langle \Psi_- | \Phi_- \rangle = \langle \Phi_- | \Psi_- \rangle &= 0 + 0 + 0 + 0 = 0 \\
\langle \Phi_+ | \Phi_- \rangle = \langle \Phi_- | \Phi_+ \rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(\frac{-1}{\sqrt{2}}) = 0
\end{aligned}$$

(b) There are many ways of doing this. You should end up with

$$\begin{aligned}
|00\rangle &= \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle) \\
|01\rangle &= \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle) \\
|10\rangle &= \frac{1}{\sqrt{2}}(|\Psi_+\rangle - |\Psi_-\rangle) \\
|11\rangle &= \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)
\end{aligned}$$

(c) Looking at the requirements, we see that the columns of $U$ should be $\Phi_+, \Psi_+, \Phi_-, \Psi_-$ and the rows should correspond to the computational basis. So,

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

(d)

$$UU^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Observe that the rows of $U^\dagger$ correspond to the Bell states coefficients $\Phi_+, \Psi_+, \Phi_-, \Psi_-$. Therefore $U^\dagger|\psi\rangle$ is the same as

$$\begin{bmatrix} \langle\Phi_+| \\ \langle\Psi_+| \\ \langle\Phi_-| \\ \langle\Psi_-| \end{bmatrix} |\psi\rangle = \begin{bmatrix} \langle\Phi_+|\psi\rangle \\ \langle\Psi_+|\psi\rangle \\ \langle\Phi_-|\psi\rangle \\ \langle\Psi_-|\psi\rangle \end{bmatrix}$$

The probabilities of the outcomes follow easily.