Quantum Computing
Problem Set 1 Solutions

1. The basic unit of quantum information is the qubit. In this problem we will discuss a single qubit system. You are given a qubit with a wave function given by $|\psi\rangle = \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$

(a) The bra corresponding to this state is:

$$\langle\psi| = \left(\frac{1}{2} - \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} + \frac{i}{2}\right)|1\rangle$$

(b)

$$\begin{aligned}\langle\psi|\psi\rangle &= \left(\frac{1}{2} - \frac{i}{2}\right)\left(\frac{1}{2} + \frac{i}{2}\right) + \left(\frac{1}{2} + \frac{i}{2}\right)\left(\frac{1}{2} - \frac{i}{2}\right) \\ &= \left(\frac{1}{2}\right)^2 - \left(\frac{i}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - \left(\frac{i}{2}\right)^2 \\ &= 1\end{aligned}$$

(c) The probabilities of obtaining the 0 or 1 states are:

$$\begin{aligned}P(0) = |v_0|^2 = v_0 v_0^* &= \left(\frac{1}{2} + \frac{i}{2}\right)\left(\frac{1}{2} - \frac{i}{2}\right) = \frac{1}{2} \\ P(1) = |v_1|^2 = v_1 v_1^* &= \left(\frac{1}{2} - \frac{i}{2}\right)\left(\frac{1}{2} + \frac{i}{2}\right) = \frac{1}{2}\end{aligned}$$

(d)

$$\begin{aligned}\langle +|\psi\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{i}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2} - \frac{i}{2}\right) \\ &= \frac{1}{2\sqrt{2}} + \frac{i}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{i}{2\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\langle -|\psi\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{i}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{2} - \frac{i}{2}\right) \\ &= \frac{1}{2\sqrt{2}} + \frac{i}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{i}{2\sqrt{2}} \\ &= \frac{i}{\sqrt{2}}\end{aligned}$$

(e) $|+\rangle$ and $|-\rangle$ are orthogonal:

$$\begin{aligned}\langle +|-\rangle &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{2} - \frac{1}{2}\\ &= 0\end{aligned}$$

and normalized:

$$\begin{aligned}\langle +|+\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\\ &= \frac{1}{2} + \frac{1}{2}\\ &= 1\\ \langle -|-\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{2} + \frac{1}{2}\\ &= 1\end{aligned}$$

(f) We're going to use the previous part. Multiplying by $\langle +|$ and $\langle -|$ respectively, we get

$$\begin{aligned}\langle +|\psi\rangle &= \langle +|\alpha|+\rangle + \langle +|\beta|-\rangle\\ &= \alpha\langle +|+\rangle + \beta\langle +|-\rangle\\ &= \alpha(1) + \beta(0)\\ &= \alpha\end{aligned}$$

$$\begin{aligned}\langle -|\psi\rangle &= \langle -|\alpha|+\rangle + \langle -|\beta|-\rangle\\ &= \alpha\langle -|+\rangle + \beta\langle -|-\rangle\\ &= \alpha(0) + \beta(1)\\ &= \beta\end{aligned}$$

(g) We already determined that $\alpha = \langle +|\psi\rangle = \frac{1}{\sqrt{2}}$ and $\beta = \langle -|\psi\rangle = \frac{i}{\sqrt{2}}$. We substitute directly in our wave equation and obtain:

$$\begin{aligned}|\psi\rangle &= \alpha|+\rangle + \beta|-\rangle\\ &= \frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle\end{aligned}$$

(h) The probabilities in the new computational basis are:

$$\begin{aligned}P(+) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2}\\ P(-) &= \frac{i}{\sqrt{2}}\left(-\frac{i}{\sqrt{2}}\right) = \frac{1}{2}\end{aligned}$$

2. We explore matrix operations on a single qubit. We have:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(a) The complex conjugate to U is

$$U^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) The adjoint to U is

$$\begin{aligned} U^\dagger &= (U^*)^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

(c) We calculate $UU^\dagger$ to show that U is indeed unitary:

$$\begin{aligned} UU^\dagger &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} + \frac{1}{2} \end{bmatrix} \\ &= I \end{aligned}$$

(d) We evolve the wave function $|\psi\rangle = \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$:

$$\begin{aligned} U|\psi\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+i}{\sqrt{2}} \\ 0 \end{bmatrix} \end{aligned}$$

(e) We have:

$$\begin{aligned} U^2 &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \end{aligned}$$

We can calculate $U^4 = UUUU = (U^2)(U^2)$ because of the associativity of matrices:

$$\begin{aligned} U^4 &= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -I \end{aligned}$$

(f) We can apply U twice to $|\psi\rangle$ as $U(U|\psi\rangle) = (UU)|\psi\rangle$:

$$\begin{aligned}
(UU)|\psi\rangle &= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1+i}{2} \end{bmatrix}
\end{aligned}$$

(g) We can express U as:

$$\begin{aligned}
U &= \frac{1}{\sqrt{2}}\sigma_0 + \frac{i}{\sqrt{2}}\sigma_1 \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}$$

Thus we have $s_0 = \frac{1}{\sqrt{2}}$ and $s_1 = \frac{i}{\sqrt{2}}$.

(h) We calculate $UZU^\dagger$ as:

$$\begin{aligned}
UZU^\dagger &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= Y
\end{aligned}$$