Quantum Computing Take Home Final

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Due: August 17, 2005

Well, here it is. The take home final (you've all been dreading?) Remember no collaboration on this final.

"I didn't fail the test, I just found 100 ways to do it wrong."
—Benjamin Franklin

"People will accept your idea much more readily if you tell them Benjamin Franklin said it first."
—David H. Commins

Problem 1: One Qubit! (30 pts)

In this problem you have been given a single qubit which has the wave function given by the ket $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$.
(a) (4 pts) What is the bra $\langle\psi|$ corresponding to this ket?
(b) (4 pts) Is this wave function normalized? This is, does $\langle\psi|\psi\rangle = 1$?
(c) (6 pts) If you measure this qubit in the computational basis, $|0\rangle$, $|1\rangle$, what are the probabilities of these two outcomes?
(d) (6 pts) Suppose we apply the unitary

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

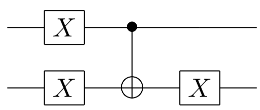
to the qubit in the wave function $|\psi\rangle$. What is the new wave function $U|\psi\rangle$?
(e) (6 pts) Recall that the Hadamard matrix is

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

If we first apply U from part (d) followed by the Hadamard matrix, this is the same as applying the unitary HU. What is the two by two matrix HU?
(f) (4 pts) Suppose we start with one qubit which has the wave function $|\psi\rangle$. Next we apply U from part (d). Then we apply the Hadamard H. What is the final qubit wave function? That is, what is $HU|\psi\rangle$?

Problem 2: Two Qubits! (40 pts)

In this problem we have been given two qubits with the wave function $|\phi\rangle = \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{2}|10\rangle$.
(a) (3 pts) What is the bra $\langle\phi|$?
(b) (4 pts) If we measure $|\phi\rangle$ in the computational basis for two qubits, what are the probabilities of the four outcomes, $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$?
(c) (5 pts) Recall that the single qubit not operator is $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the single qubit identity operator is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Write out the two qubit unitary matrix $I \otimes X$ in the computational basis.
(d) (5 pts) Write out the two qubit unitary matrix $X \otimes X$ in the computational basis.
(e) (5 pts) Suppose we apply the unitary matrix $X \otimes X$ to $|\phi\rangle$. What is the resulting two qubit state $(X \otimes X)|\phi\rangle$?
(f) (5 pts) Suppose that we feed $|\phi\rangle$ into the following circuit



What is the resulting two qubit wave function?

(g) (3 pts) Return now to $|\phi\rangle$. Suppose we are given two qubits with this wave function and we measure the first of these two qubits in the computational basis, $|0\rangle$, $|1\rangle$. What are the probabilities of these two outcomes?

(h) (3 pts) Recall that the Bell basis are given by the four two qubit kets

$$\begin{aligned} |\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \\ |\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \end{aligned}$$

Suppose that we measure $|\phi\rangle$ in the Bell basis. What are the probabilities of the four outcomes $|\Phi_+\rangle$, $|\Phi_-\rangle$, $|\Psi_+\rangle$, and $|\Psi_-\rangle$?

(i) (7 pts) Suppose in addition to $|\phi\rangle$, we have a third qubit whose wave function is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, i.e. we start with $|\phi\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Suppose you now measure the first and third qubits of this system in the Bell basis. After this measurement, the wave function of the second qubit will be separable from the wave function of the first and the third qubit. Suppose that you got the outcome corresponding to $|\Phi_+\rangle$: what would the wave function of the second qubit then be?

Problem 3: Outer Products (10 pts)

(a) (3 pts) Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Write out the two by two matrix $|+\rangle\langle+|$ in the computational basis, $|0\rangle$, $|1\rangle$.

(b) (4 pts) Let $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Suppose that we are given two qubits whose wave function is $\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|11\rangle$. If we measure the first of these qubits in the $|+i\rangle$, $|-i\rangle$ basis, what are the probabilities of these two outcomes? What is the wave function of these two qubits after the measurement for each of these two possible outcomes?

(c) (3 pts) Define the four dimensional matrix $V = |00\rangle\langle 00| + e^{\frac{2\pi i}{3}}|01\rangle\langle 01| + \frac{1}{\sqrt{2}}(|10\rangle\langle 10| + i|11\rangle\langle 10| + i|10\rangle\langle 11| + |11\rangle\langle 11|)$. Is this matrix unitary? That is does $VV^\dagger = I$?

Problem 4: n Qubits! (10 pts)

In this problem we will deal with n qubits.

(a) (4 pts) Suppose that we have n qubits which have the wave function $|0\rangle = |0, 0, \ldots, 0\rangle$. If we now apply the n qubit Pauli X to these n qubits, $X^{\otimes n}$ (where X is defined Problem 2), what is the resulting n qubit state? Your answer should be a single computational basis state.

(b) (4 pts) Recall the definition of the Hadamard from Problem 1. Suppose we apply the n qubit Hadamard, $H^{\otimes n}$ to $(X^{\otimes n})|0\rangle$. (H is defined in Problem 1.) What is the resulting n qubit wave function? Express it as a sum over computational basis kets, i.e. in the form $\sum_{x=0}^{2^n-1} a_x|x\rangle$, where a_x is some function of x which you must find a formula for.

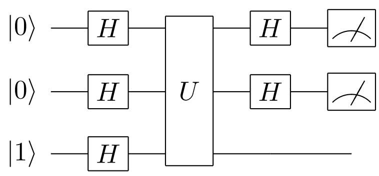
(c) (2 pts) $H^{\otimes n}X^{\otimes n}H^{\otimes n}$ can be expressed as $A^{\otimes n}$. What is A?

Problem 5: An Algorithm Problem (10 pts)

Let $f(x_1, x_2) = x_1 \oplus x_2$.

(a) (2 pts) Recall (from the Deutsch-Jozsa problem) that a constant function is one that is the same on all inputs and a balanced function is one that is equal to 0 on half of the inputs and 1 on the other half of the inputs. Is $f(x_1, x_2)$ constant or balanced?

(b) (2 pts) The Deustch-Jozsa algorithm for three qubits is given by the circuit



Recall that the meters mean measurement in the computational basis. If we are running this circuit for the f

specified above, then U is equal to

$$U = \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}\sum_{y=0}^{1} |x_1, x_2, y \oplus f(x_1, x_2)\rangle\langle x_1, x_2, y|$$

What is $U\frac{1}{2\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes (|0\rangle - |1\rangle)$? Express your result as a separable wave function between the wave function of the first two qubits and the wave function of the third qubit.

(c) (3 pts) In the above circuit what is the probability that the two measurements both result in the outcome $|0\rangle$ (i.e. the measurement of the two qubits yields $|00\rangle$)?

(d) (3 pts) In the above circuit what are the probabilities of all four measurement outcomes ($|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$)? Explain how the result of this measurement can be used to determine whether f was constant or balanced a la the Deutsch-Jozsa algorithm.

Extra Credit Problem: The GHZ Problem (9 pts)

In this problem we will discuss entanglement in a three party setting. Let's label these different parties, Alice, Bob, and Charlie. Suppose that Alice, Bob, and Charlie are locked away in three different sound proof rooms and so are not able to talk to each other. In order to gain their release from being locked up, the warden has designed a game which they will play. In this game, the warden gives Alice, Bob, and Charlie each a slip of paper with either X or Y written on it. The warden tells all three parties that he has either given all of the parties slips with X on the paper, or he has given two of parties slips with Ys on them and the remaining party has a slip with X on it. The players only know what is on their slip of paper, and not what is on the other two parties' slips of paper. To be clear, if we label what is on the slips of paper given to Alice, Bob, and Charlie as (A, B, C), respectively, then the warden gives them either (X, X, X), (X, Y, Y), (Y, X, Y), or (Y, Y, X). Now in order to gain their release, the warden tells the parties they must complete the following task. The warden will come around, after he has given the the slips of paper, and ask each of the parties to say either $+1$ or -1. The warden will then take these three numbers and multiply them together. The parties win under the following condition: if the warden gave all three players slips with X on them, then the product of all of their numbers must be $+1$ for them to win and if the warden gave two of the players slips with Ys on them, then the product of all of their numbers must be -1. They lose in all other cases.

(a) (3 pts) Suppose that Alice, Bob, and Charlie do not share any entangled quantum states. Prove that it is impossible for Alice, Bob, and Charlie to always (i.e. no matter what the warden gives them ((X, X, X), (X, Y, Y), (Y, X, Y), or (Y, Y, X))) win this game.

(b) (3 pts) Now suppose that Alice, Bob, and Charlie, before they were locked away, took three qubits and put them into the GHZ wave function, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. When Alice, Bob, and Charlie were separated, they each kept one of the qubits from this three qubit system. Now (amazingly!) it will be possible for them to *always* win the game the warden has set up. Each party, if they are given a slip with X on it, they will measure their qubit in the $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle$, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle$ basis, and if they are given a slip with Y on it, they will measure their qubit in the $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle$, $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ basis. For the four possible combinations of the slips of paper, calculate the probabilities of all eight outcomes for their measurements. (Yes, your answer will have $4 \times 8 = 32$ different numbers. That's why this is extra credit!)

(c) (3 pts) Deduce from the above measurement results how the parties can use the GHZ state and their measurements to *always* win the game the warden has set up for them.