CSEP 590TU
Sample Final Questions

1. For each of the following questions answer true or false and JUSTIFY your answer.

   (a) If L is Turing recognizable then there is a Turing machine that generates L in lexicographic order.

   (b) If A is $\mathcal{NP}$-complete and $A \leq_P B$ then B is $\mathcal{NP}$-complete.

   (c) $SPACE(\log n) \subseteq \mathcal{P}$.

   (d) Every regular language is in $SPACE(c)$ for any $c \geq 1$.

   (e) The language $\{\langle M \rangle \mid M \text{ accepts } \langle M \rangle\}$ is Turing-recognizable.

2. Let $L = \{\langle M \rangle \mid M \text{ accepts 1 but does not accept 0}\}$.
   In this question you will show that L is neither Turing recognizable nor co-Turing recognizable by showing that $A \leq_m L$ and $\overline{A} \leq_m L$ for some non-recursive language A.

   (a) Show that $E_{TM} \leq_m L$.

   (b) Show that $\overline{E_{TM}} \leq_m L$.

3. For $A, B \subseteq \Sigma^*$, we say that A is *linear-time mapping reducible* to B, written $A \leq_m^{lin} B$ if and only if there is a function $f : \Sigma^* \rightarrow \Sigma^*$ which is computable in linear time, (i.e. $f(x)$ is computable in time at most $c|x|$ for some constant c) such that for all $x \in \Sigma^*$, $x \in A \Leftrightarrow f(x) \in B$.

   (a) Prove that if $A \leq_m^{lin} B$ and $B \leq_m^{lin} C$ then $A \leq_m^{lin} C$.

   (b) Prove that if $B \in TIME(n^k)$ and $A \leq_m^{lin} B$ then $B \in TIME(n^k)$.

   (c) It is known that for all k, there are languages in P that are not in $TIME(n^k)$; i.e. there is no single fixed time upper bound for problems in P. Use part (b) and this fact to prove that there are no problems $B \in P$ that satisfy $A \leq_m^{lin} B$ for all $A \in P$.
   (Saying this in a long-winded way 'there are no problems that are complete for P under linear-time mapping reductions.')

4. Prove that:

   (a) If K and L are in $\mathcal{NP}$ then $KL = \{x \mid x = yz \text{ for some } y, z \text{ with } y \in K \text{ and } z \in L\}$ is in $\mathcal{NP}$.

   (b) If L is in $\mathcal{NP}$ then L^* is in $\mathcal{NP}$ where

   $$L^* = \{x \mid x = w_1 w_2 \cdots w_k \text{ for some } k \text{ where } w_1 \in L, w_2 \in L, \ldots, w_k \in L\}.$$

   (That is L^* consists of any string that can be split into pieces each of which is in L.)

5. Do ONE of the following two questions.

   (a) Prove that the SET COVER problem defined below is $\mathcal{NP}$-complete:
   Given a collection of sets $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$ and an integer k, is there a collection of k of these sets whose union is all of $\{1, \ldots, n\}$? That is, do there exist $i_1, \ldots, i_k \leq m$ such that $\bigcup_{j \leq k} S_{i_j} = \{1, \ldots, n\}$?
   (Hint: Use the fact that DOMINATING SET is $\mathcal{NP}$-complete.)
   NOTE: Before worrying about exactly HOW to show this, make sure that you describe exactly WHAT you are going to try to do.

(b) Prove that the ZERO COST SIMPLE CYCLE problem defined below is $\mathcal{NP}$-complete:
Given a *directed* graph $G = (V, E)$ of n vertices with integer weights on its edges (possibly negative), is there a simple directed cycle of length > 0 in G with total weight 0?
(A cycle is *simple* if every vertex on the cycle appears exactly once. The total weight of a cycle is the sum of the weights on its edges.)
Hint: Use the fact that HAMILTONIAN CIRCUIT is $\mathcal{NP}$-complete.
NOTE: Before worrying about exactly HOW to show this, make sure that you describe exactly WHAT you are going to try to do.