

# Planar Geometry

CSE P576

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These slides were developed by Dr. Matthew Brown for CSEP576 Spring 2020 and adapted (slightly) for Fall 2021  
credit → Matt  
blame → Vitaly

# Image Alignment

- Aim: warp our images together using a 2D transformation



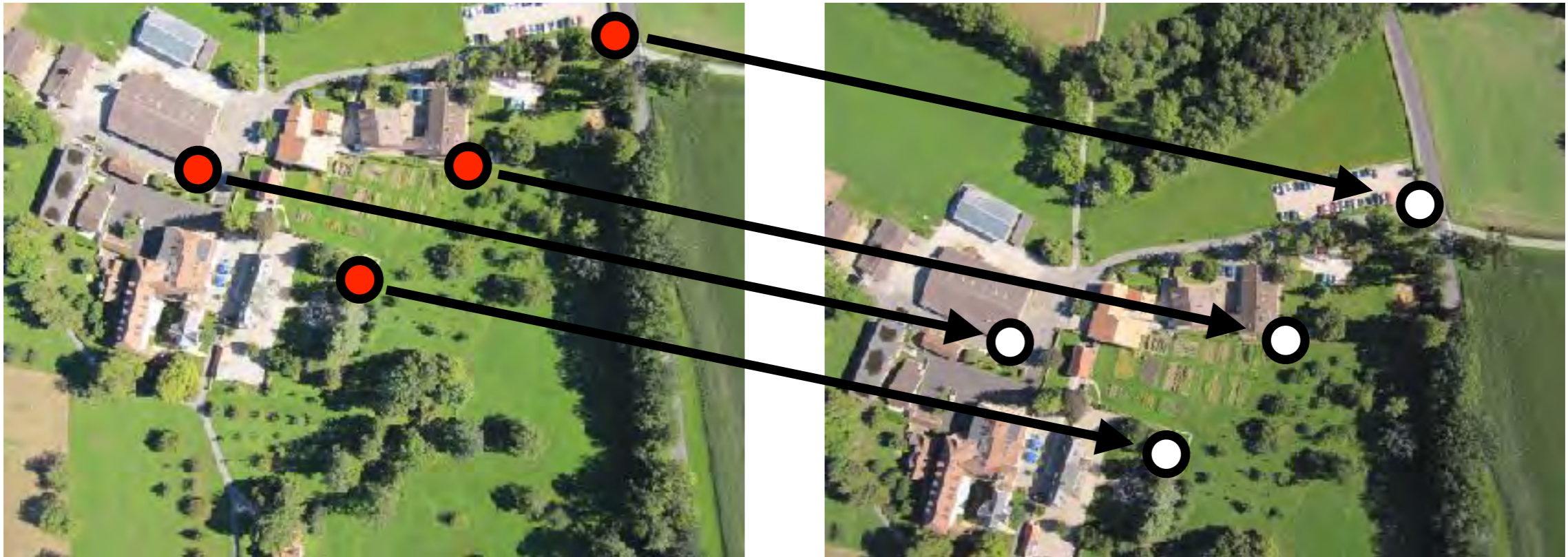
# Image Alignment

- Aim: warp our images together using a 2D transformation



# Image Alignment

- Find corresponding (matching) points between the images



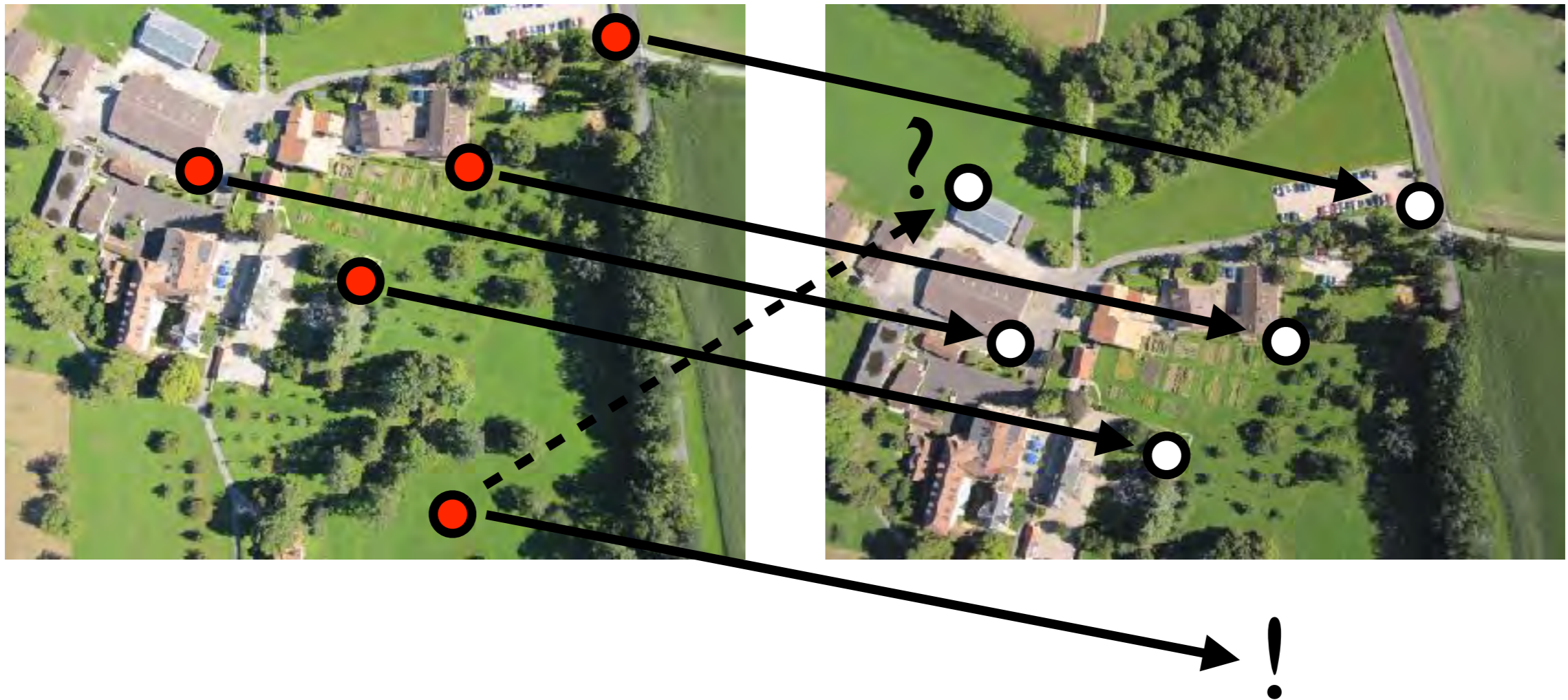
# Image Alignment

- Compute the transformation to align the points



# Image Alignment

- We can also use this transformation to reject outliers



# Image Alignment

- We can also use this transformation to reject outliers



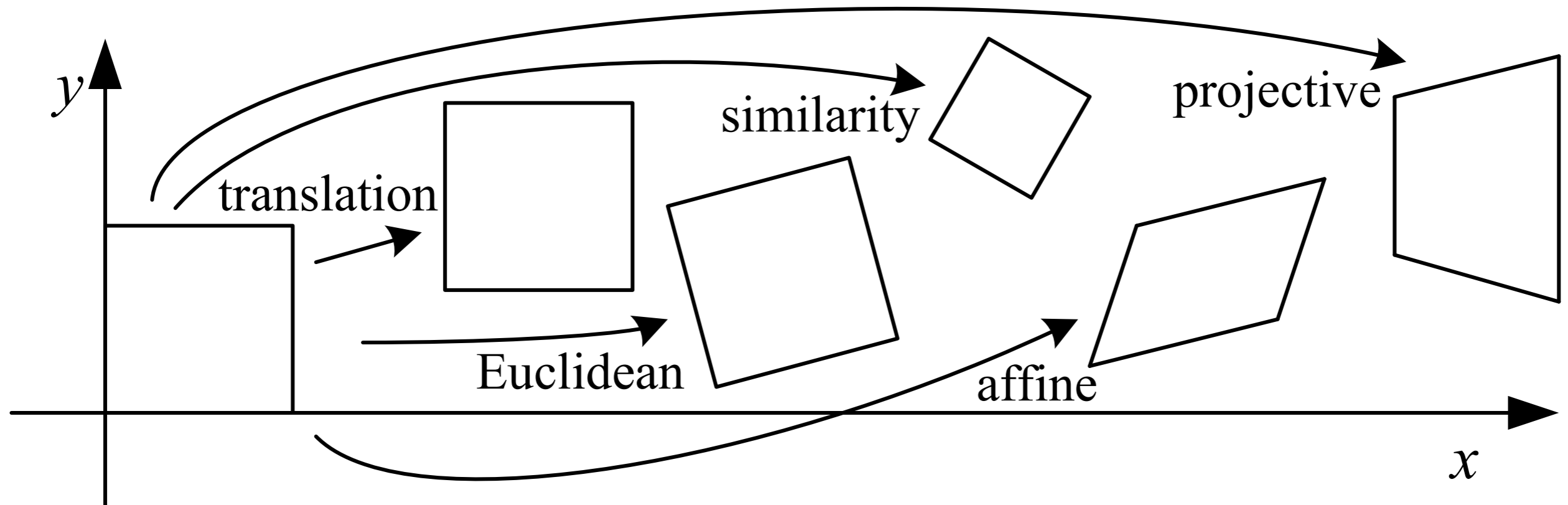
# Planar Geometry

- 2D Linear + Projective transformations
  - Euclidean, Similarity, Affine, Homography
- Linear + Projective Cameras
  - Viewing a plane, rotating about a point



# 2D Transformations

- We will look at a family that can be represented by 3x3 matrices



This group represents perspective projections of **planar surfaces** in the world

# Affine Transformations

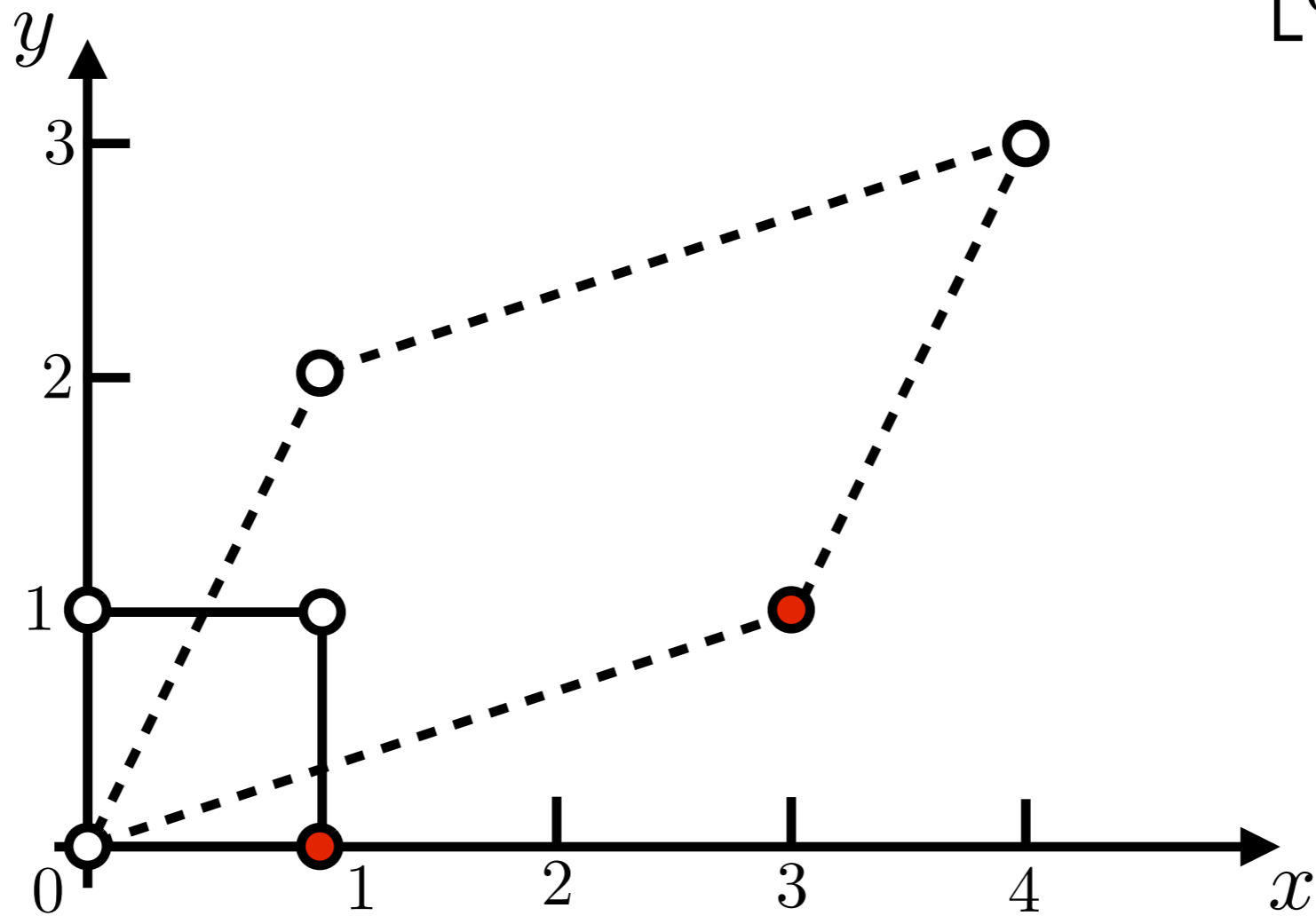
- Transformed points are a linear function of the input points

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

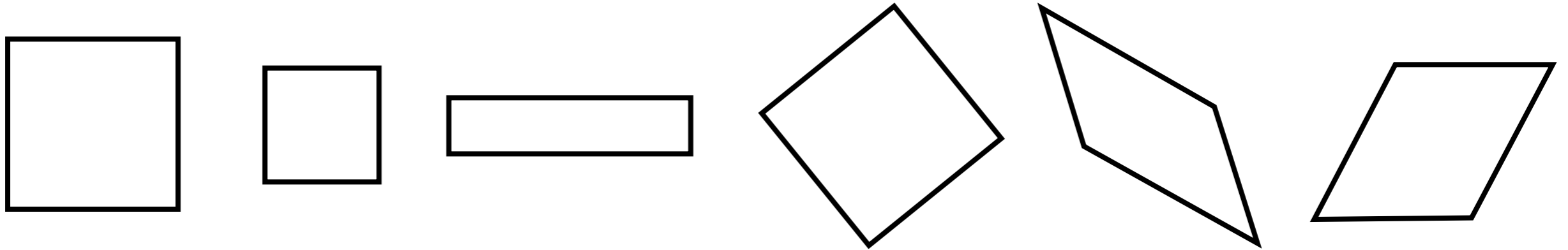
- This can be written as a single matrix multiplication

# Linear Transformations

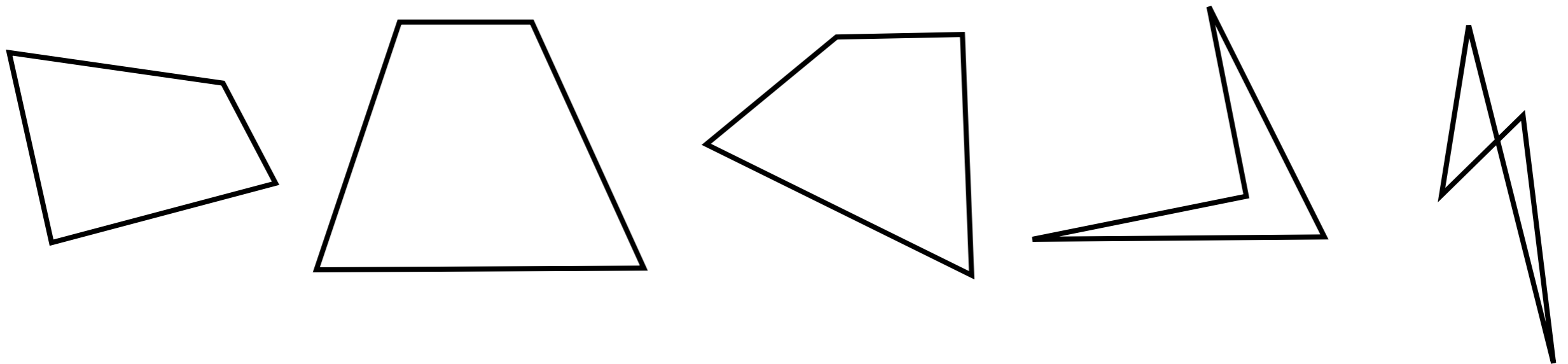
- Consider the action of the unit square under  $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



# Linear Transform Examples



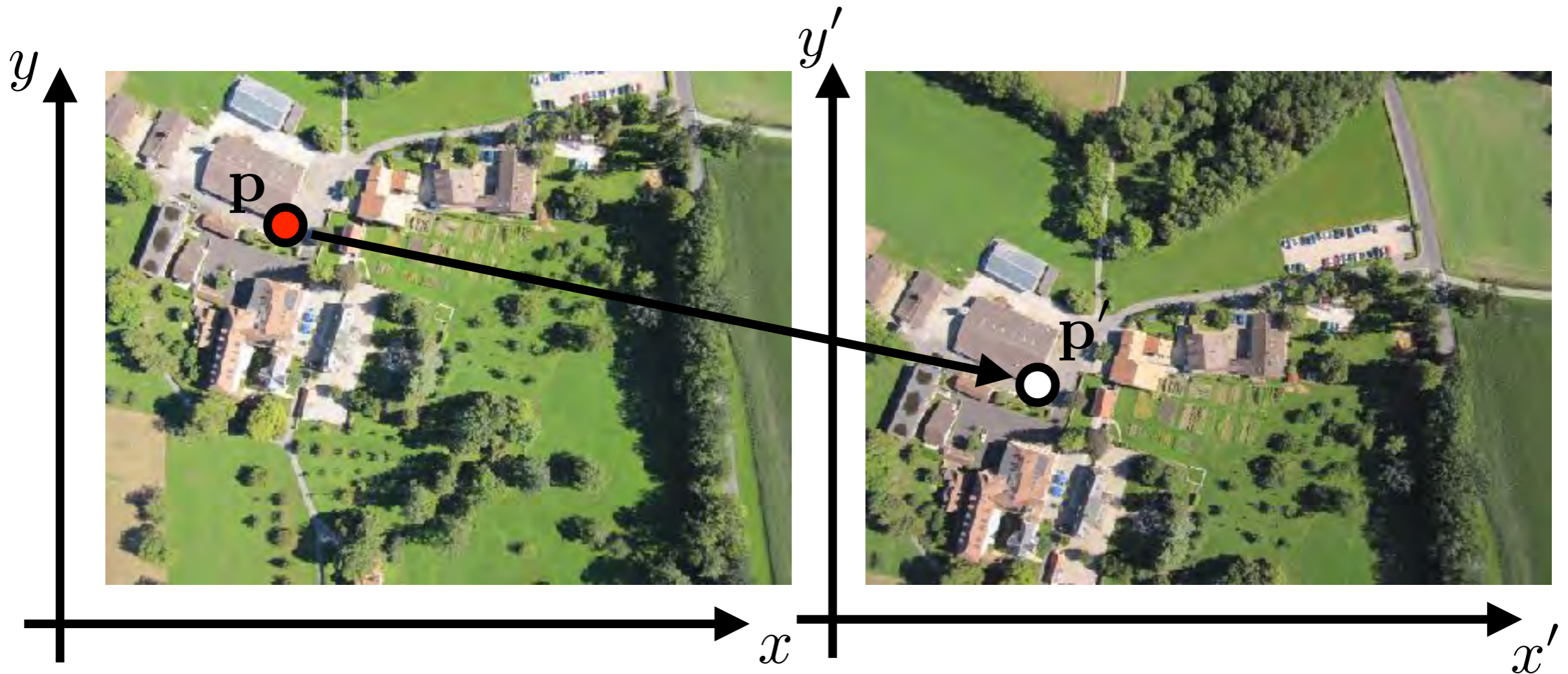
Translation, rotation, scale, shear (parallel lines preserved)



These transforms are not affine (parallel lines not preserved)

# Linear Transformations

- Consider a single point correspondence



$$\begin{bmatrix} x'_1 \\ y'_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$



How many points are needed to solve for  $\mathbf{a}$ ?

# Computing Affine Transforms

- Lets compute an affine transform from correspondences:

$$\begin{bmatrix} x'_1 \\ y'_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

- Re-arrange unknowns into a vector

# Computing Affine Transforms

- Linear system in the unknown parameters  $\mathbf{a}$

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \end{bmatrix}$$

- Of the form

$$\mathbf{M}\mathbf{a} = \mathbf{y}$$

Solve for  $\mathbf{a}$  using Gaussian Elimination

# Computing Affine Transforms

- We can now map any other points between the two images



$$\begin{bmatrix} x'_1 \\ y'_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$



# Computing Affine Transforms

- Or resample one image in the coordinate system of the other

This allows us to “stitch”  
the two images

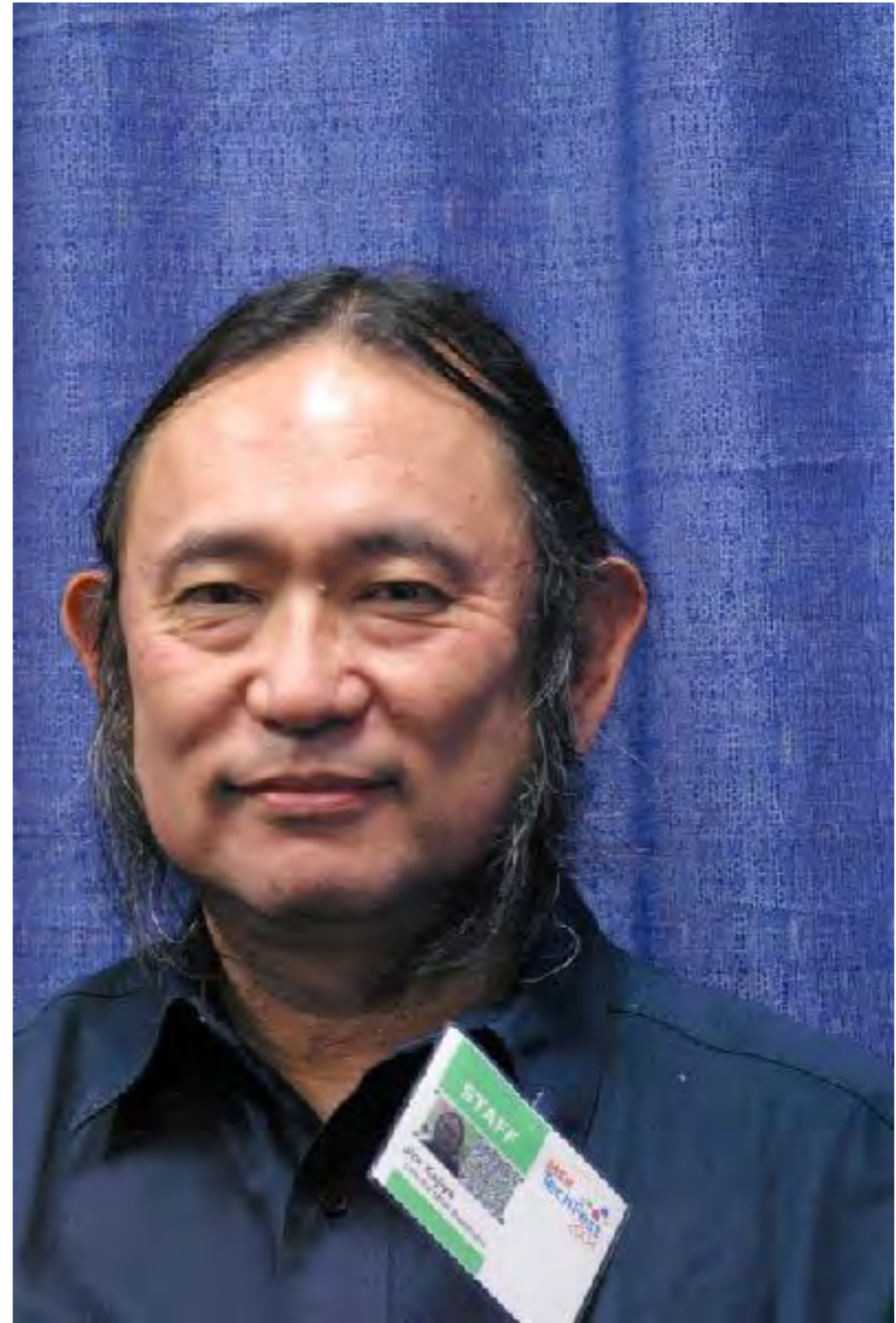
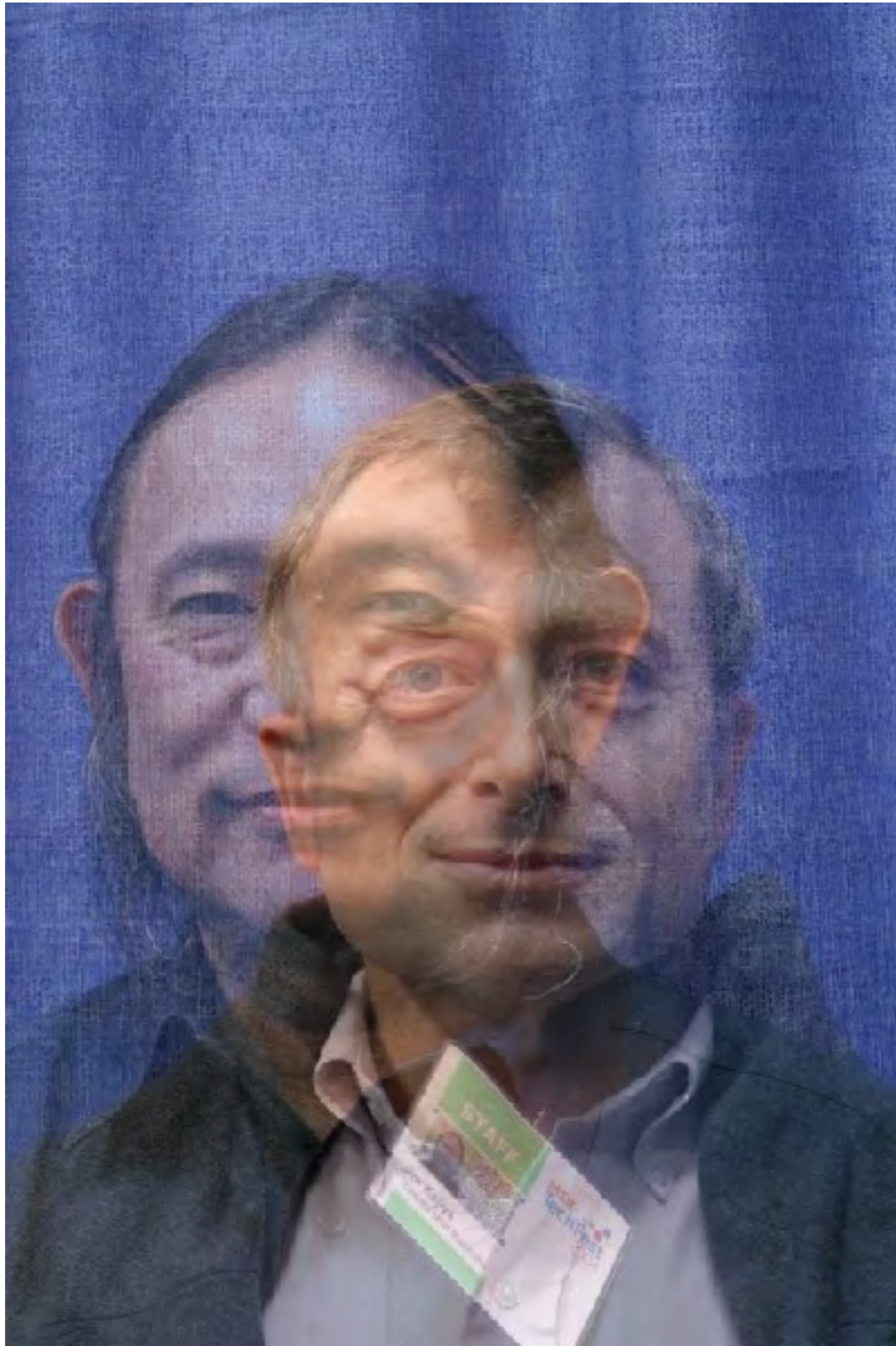


# Linear Transformations

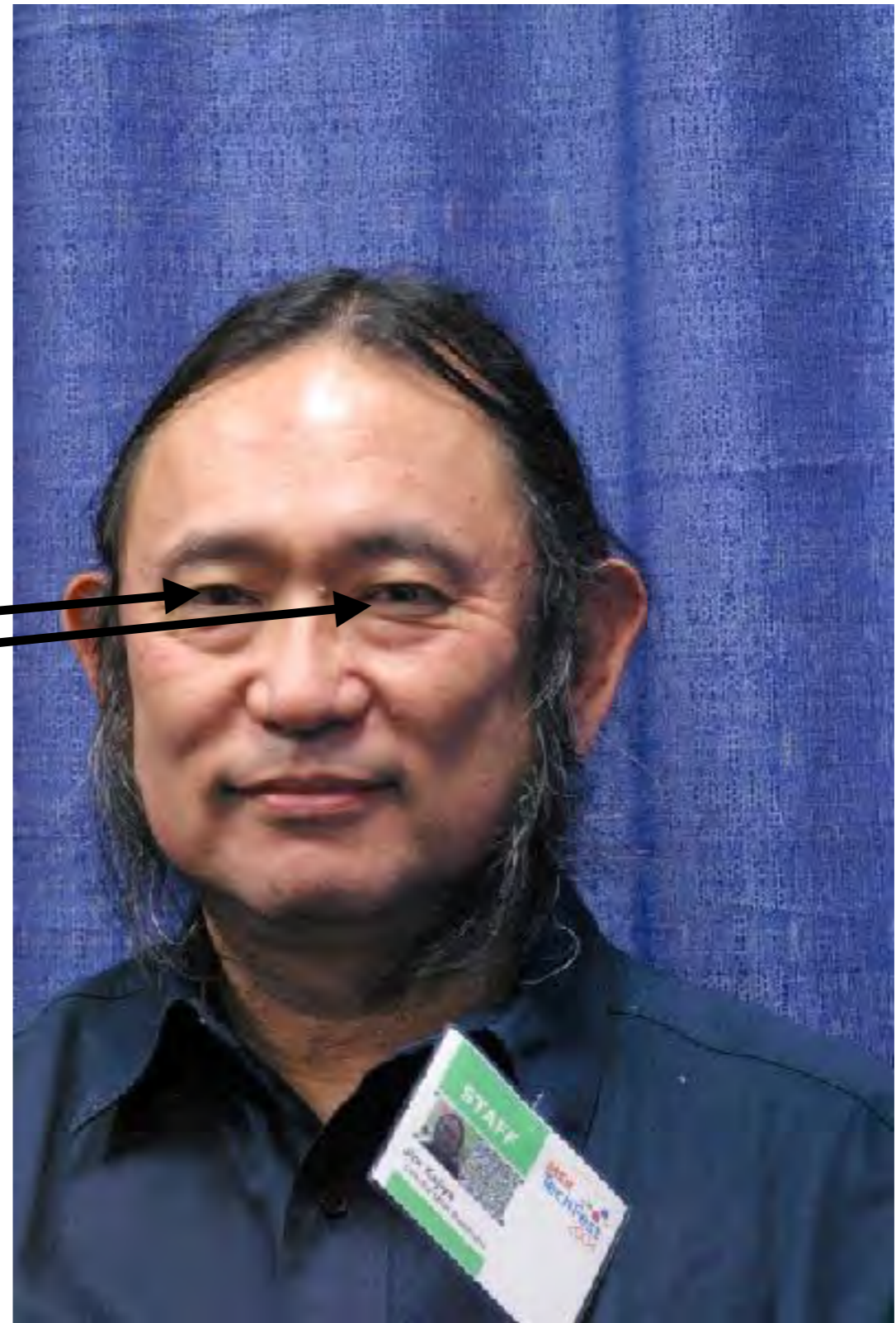
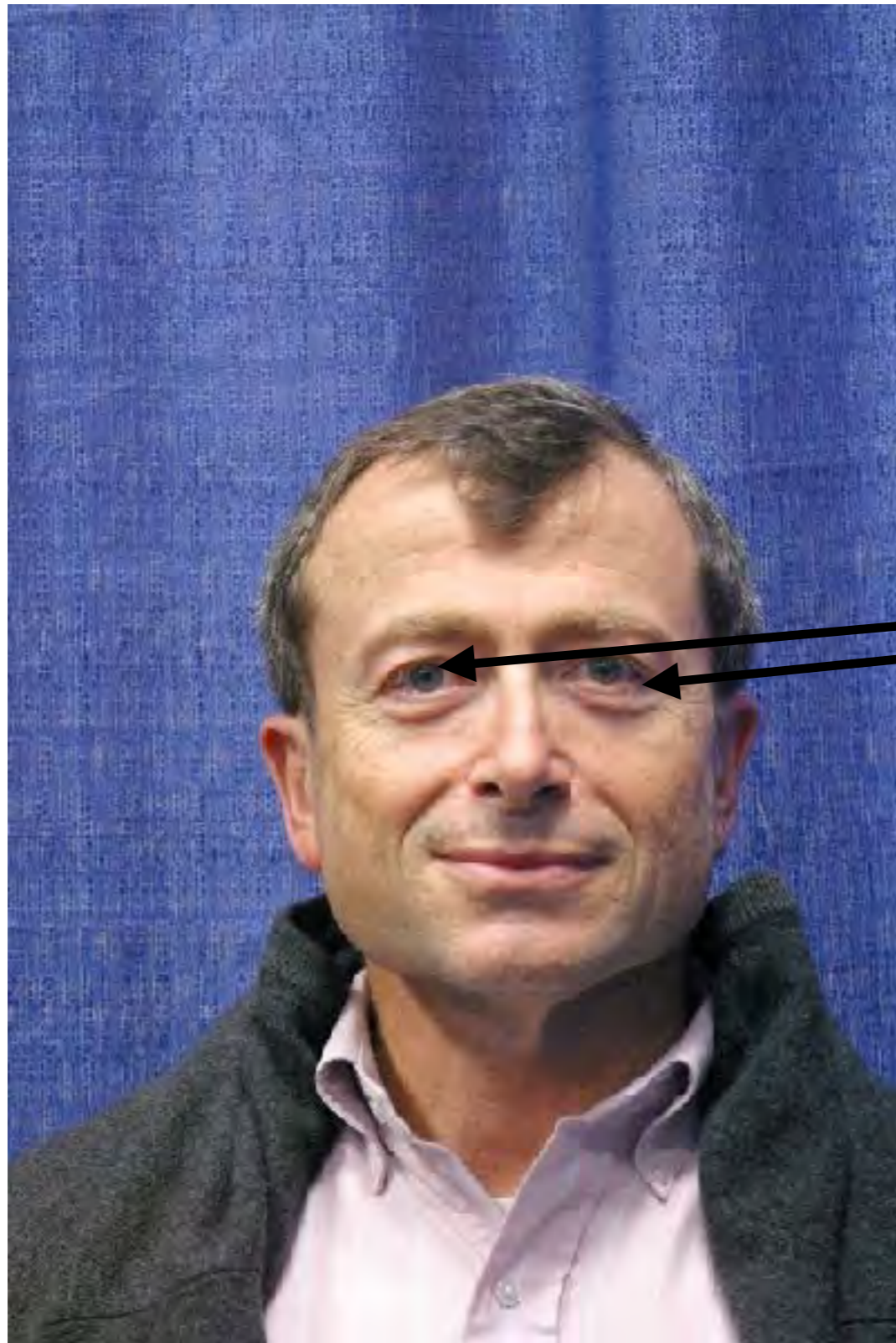
- Other linear transforms are special cases of affine

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

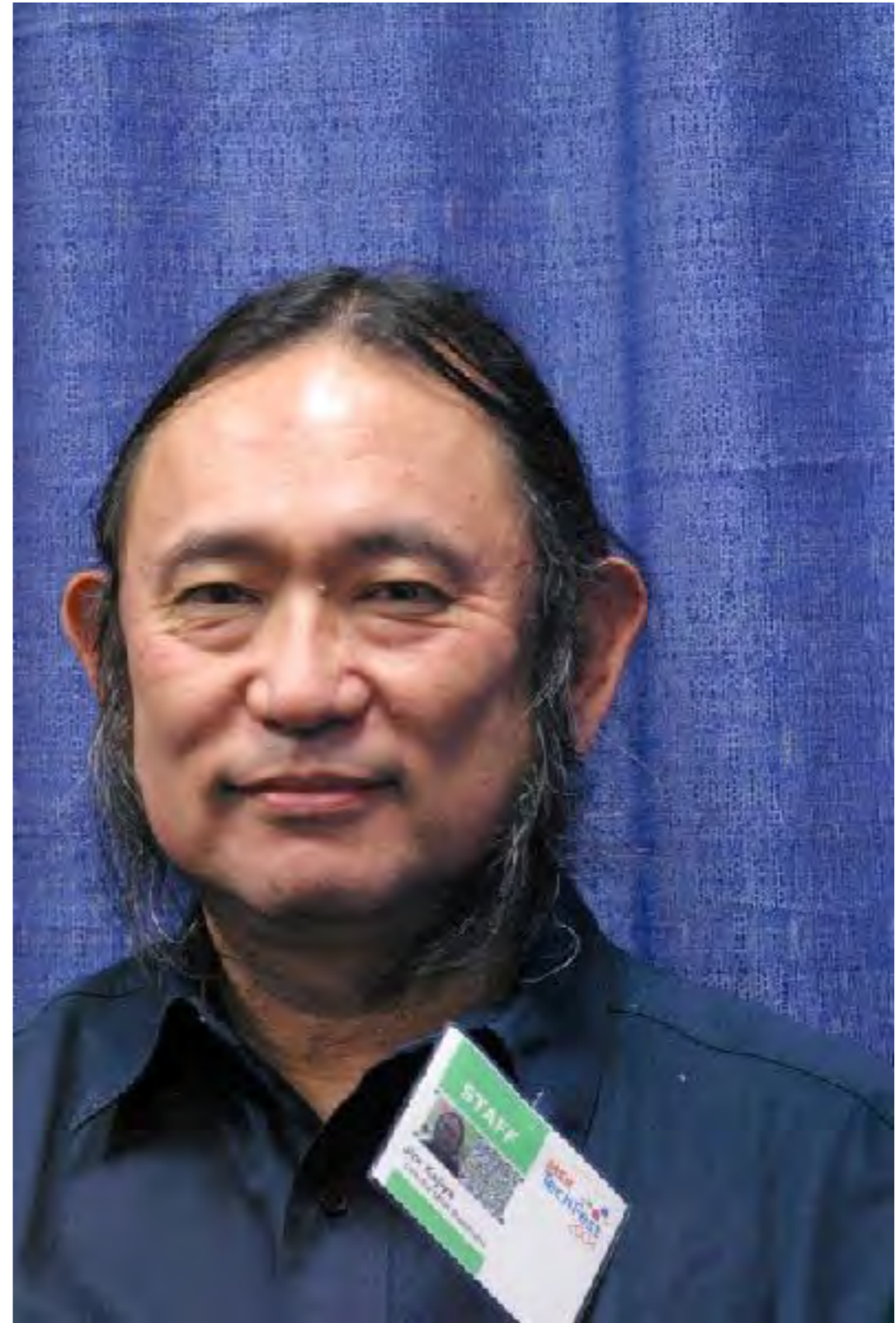
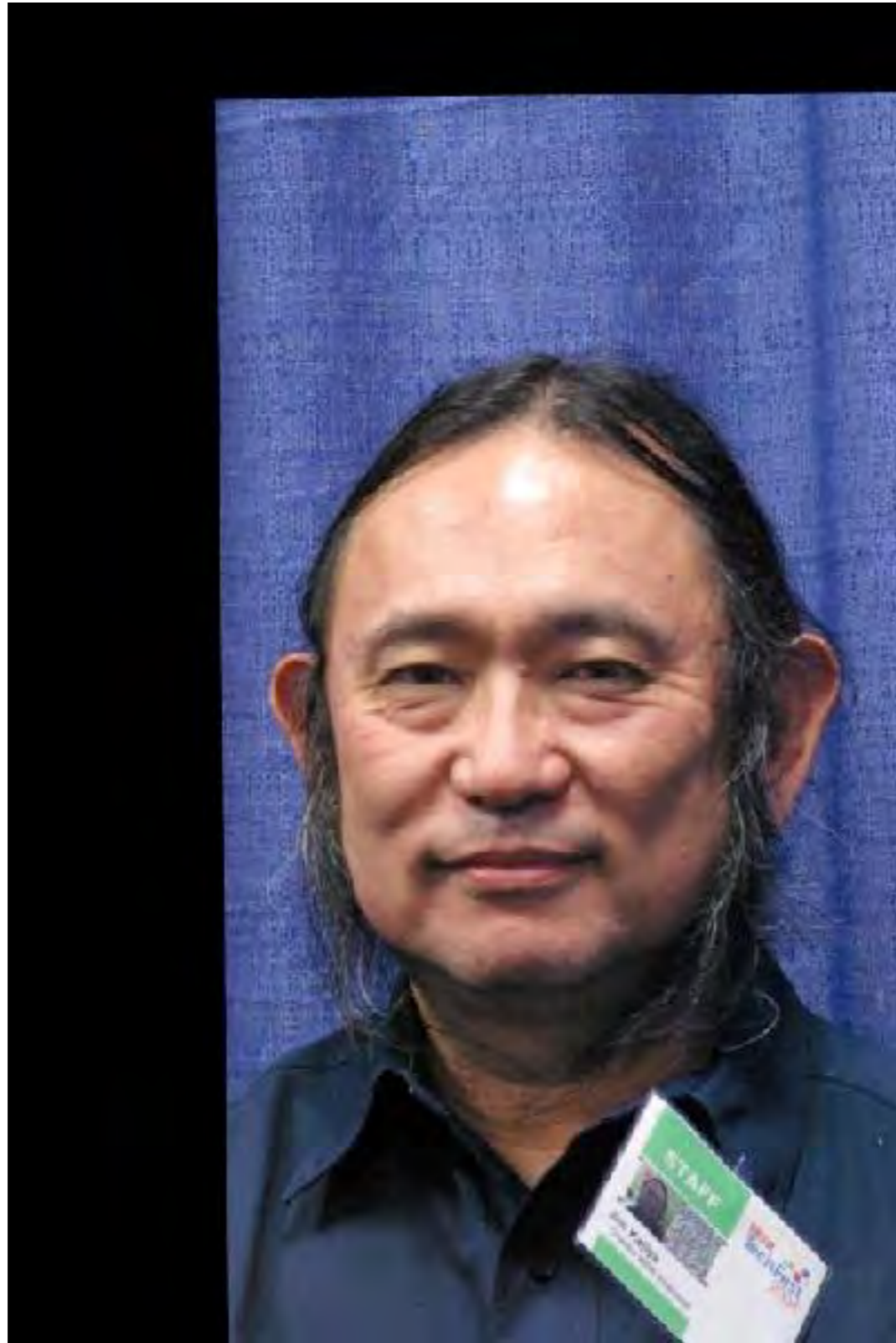
# Face Alignment



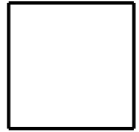
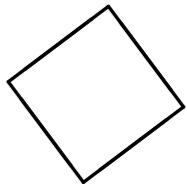
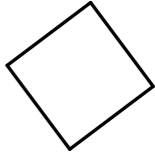
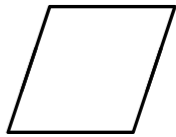
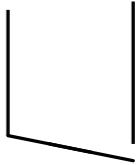
# Face Alignment



# Face Alignment



# 2D Transformations

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} &   & \mathbf{t} \end{bmatrix}_{2 \times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} &   & \mathbf{t} \end{bmatrix}_{2 \times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} &   & \mathbf{t} \end{bmatrix}_{2 \times 3}$	4	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2 \times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3 \times 3}$	8	straight lines	

# Projective Transformation

- General 3x3 matrix transformation (note need scale factor)

$$s \begin{bmatrix} x'_1 \\ y'_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

# Project 2



- Try out the **Image Warping Test** section in Project 2, particularly similarity, affine and projective transforms. You can also try warping with the inverse transform, e.g., using `P=np.linalg.inv(P)`



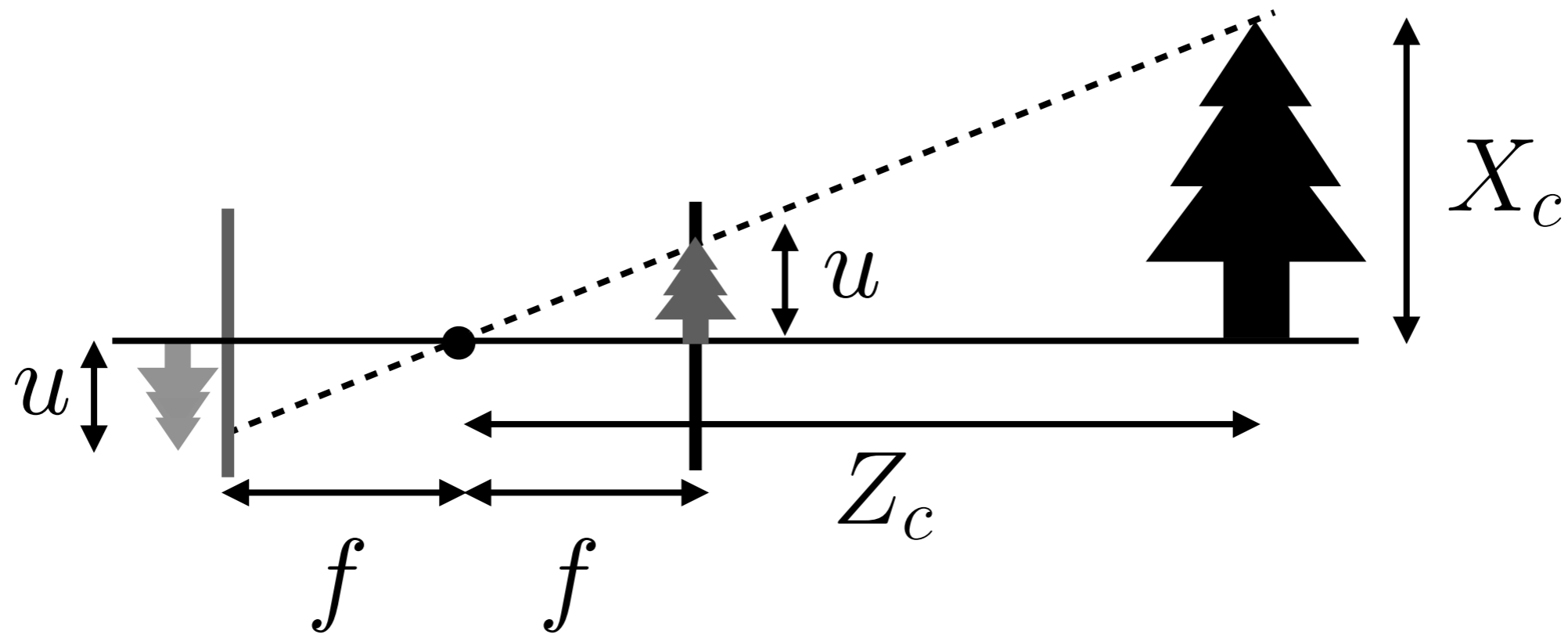
# Camera Models + Geometry



- Pinhole camera, rigid body coordinate transforms
- Perspective, projective, linear/affine models
- Properties of cameras: viewing parallel lines, viewing a scene plane, rotating about a point

# Pinhole Camera

- Put the projection plane in front to avoid the  $180^\circ$  rotation

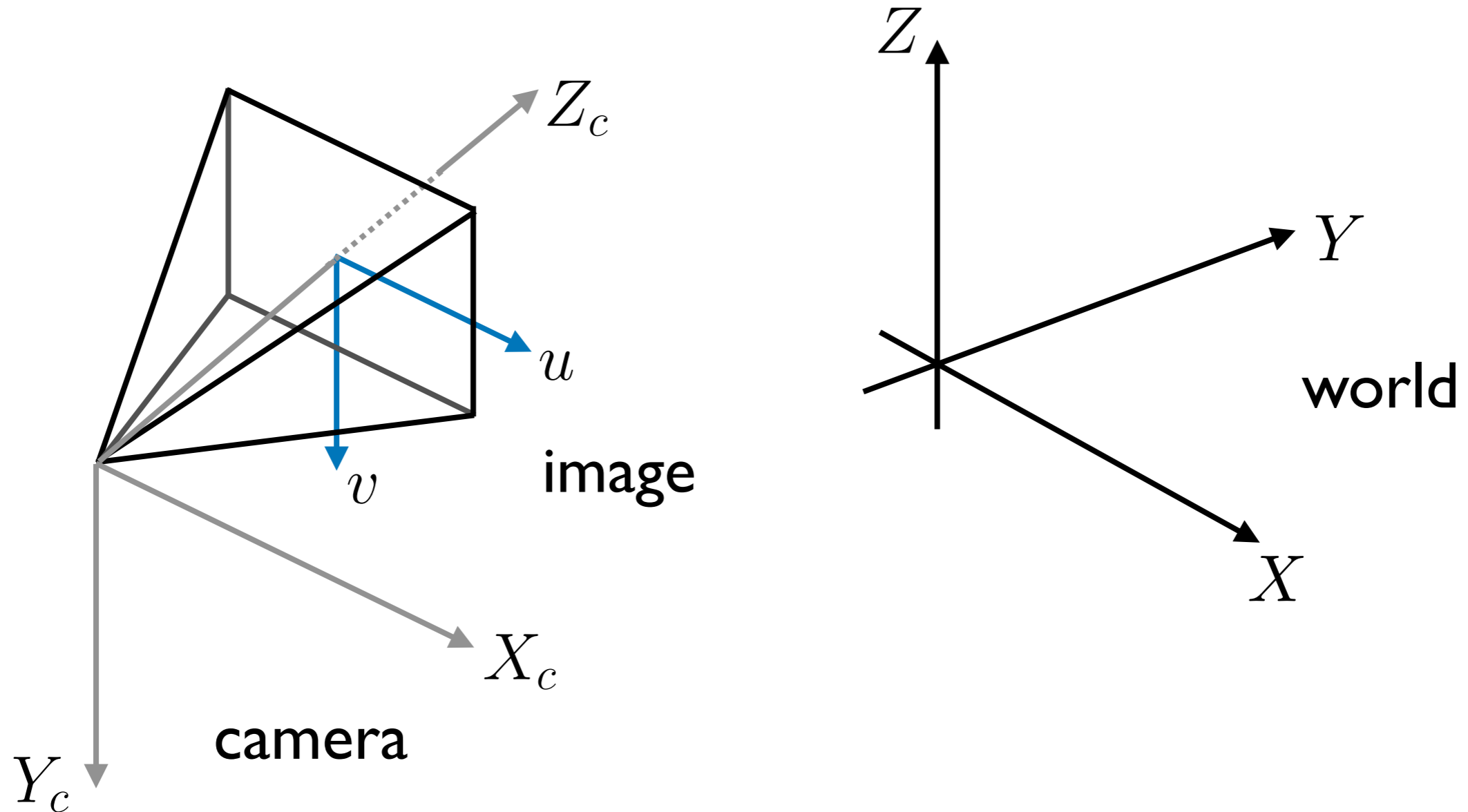


$$u = f X_c / Z_c$$
$$v = f Y_c / Z_c$$
$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}$$

- Note that  $X_c Y_c Z_c$  are **camera coordinates**

# Perspective Camera

- Transform world to camera, to image coordinates



# Projective Camera

- Perspective camera equation

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Multiply and drop constraints to get a general 3x4 matrix

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

This is called a **projective camera**



How many degrees of freedom do these 2 models have?

# Linear Camera

- Zero out bottom row to eliminate perspective division

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Linear a.k.a. affine camera

# Linear vs Projective Cameras

- Consider a linear / affine camera viewing parallel world lines

Consider parallel lines in 3D (rays)

$$l_i = \underbrace{r_i}_{\text{point}} + \lambda \underbrace{t}_{\text{direction}} \quad \lambda \in (-\infty, \infty)$$

$$\lambda \in [0, \infty)$$

LINEAR CAMERA:  $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 2 \times 3 & 2 \times 1 \\ A & a \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \Rightarrow l_i \mapsto A \cdot r_i + a + \lambda \underbrace{At}_{\text{new direction}}$

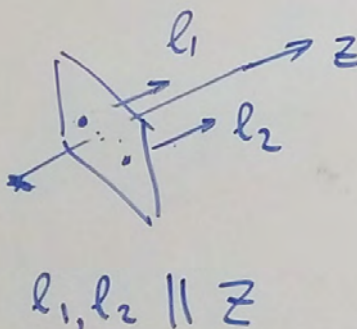
perspective camera: Example

imaged line

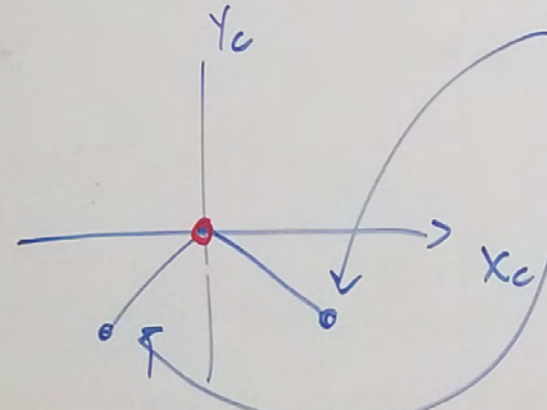
new direction

Same for all imaged lines

∴ parallel lines stay parallel



$l_1, l_2 \parallel Z$



$X_c, Y_c, Z_c$

$[1, -1, 1]$

$[-1, -1, 1]$

since  $u = f \frac{X_c}{Z_c} \rightarrow [f, -f]$

$v = f \frac{Y_c}{Z_c} \rightarrow [-f, f]$

HOWEVER,  $(u, v) \xrightarrow{Z_c \rightarrow \infty} (0, 0)$

# Linear vs Projective Cameras



Parallelism preserved if depth variation in scene  $\ll$  depth of scene

# More on camera rays and vanishing points

COMPUTING RAYS FROM AN IMAGE POINT INTO THE 3D WORLD

① LET A RAY (IN 3D) BE DEFINED BY  $d$  (direction)

POINTS ON THE RAY ARE  $\lambda \cdot d$  in 3D or  $\begin{bmatrix} \lambda d \\ 1 \end{bmatrix}$  in  $P^3$  (projective space)

$\downarrow$  map  $h$

$X = K \begin{bmatrix} I & | & 0 \end{bmatrix} \begin{bmatrix} \lambda d \\ 1 \end{bmatrix} \approx Kd$  (dropping the  $\lambda$  scale factor)

$\uparrow$  projective equivalence

$\therefore d = K^{-1} \cdot X$

$\uparrow$  not necessarily a unit vector

② VANISHING POINT

CONSIDER A RAY IN 3D FROM A POINT  $A$  IN DIRECTION  $d$

$X(\lambda) = A + \lambda \cdot \begin{bmatrix} d \\ 0 \end{bmatrix} \quad \lambda \in [0, \infty)$   $\Rightarrow$   $\underbrace{X(\lambda)}_{\text{image}} = \underbrace{K \begin{bmatrix} I & | & 0 \end{bmatrix}}_P (A + \lambda \begin{bmatrix} d \\ 0 \end{bmatrix}) \Rightarrow$

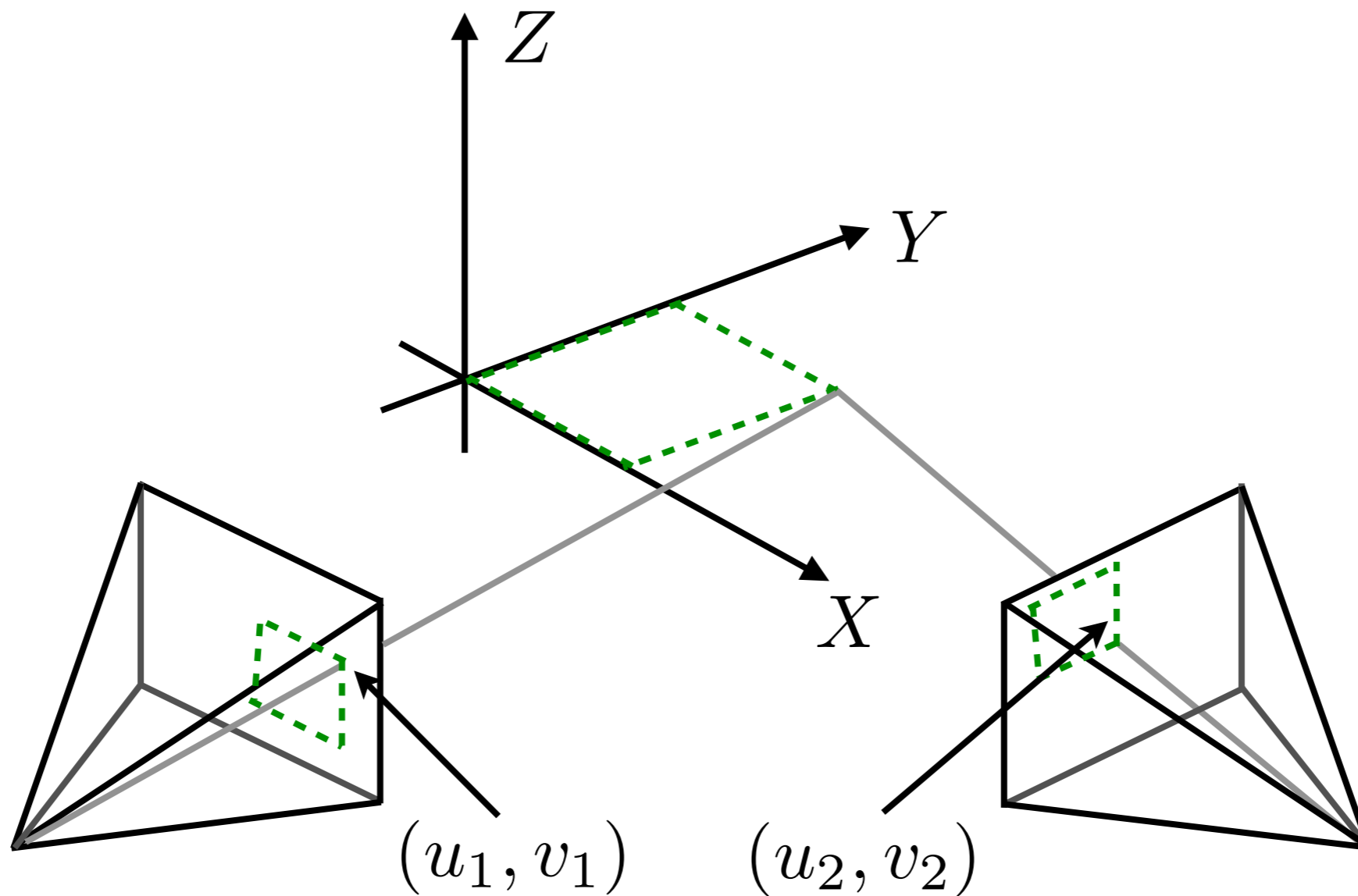
$\lim_{\lambda \rightarrow \infty} X(\lambda) = \lim_{\lambda \rightarrow \infty} (a + \underbrace{\lambda \cdot K \cdot d}_{\text{dominant term}}) \approx \boxed{K \cdot d}$  VANISHING POINT  $\underbrace{a + \lambda Kd}_{P \cdot A}$

projective space equivalence



# Viewing a Plane

- Consider a pair of cameras viewing a plane



Without loss of generality, we can make it the world plane  $Z=0$  <sub>32</sub>

# Viewing a Plane

- Viewing the plane  $Z=0$  with projective + linear cameras

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{14} \\ p_{21} & p_{22} & p_{24} \\ p_{31} & p_{32} & p_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

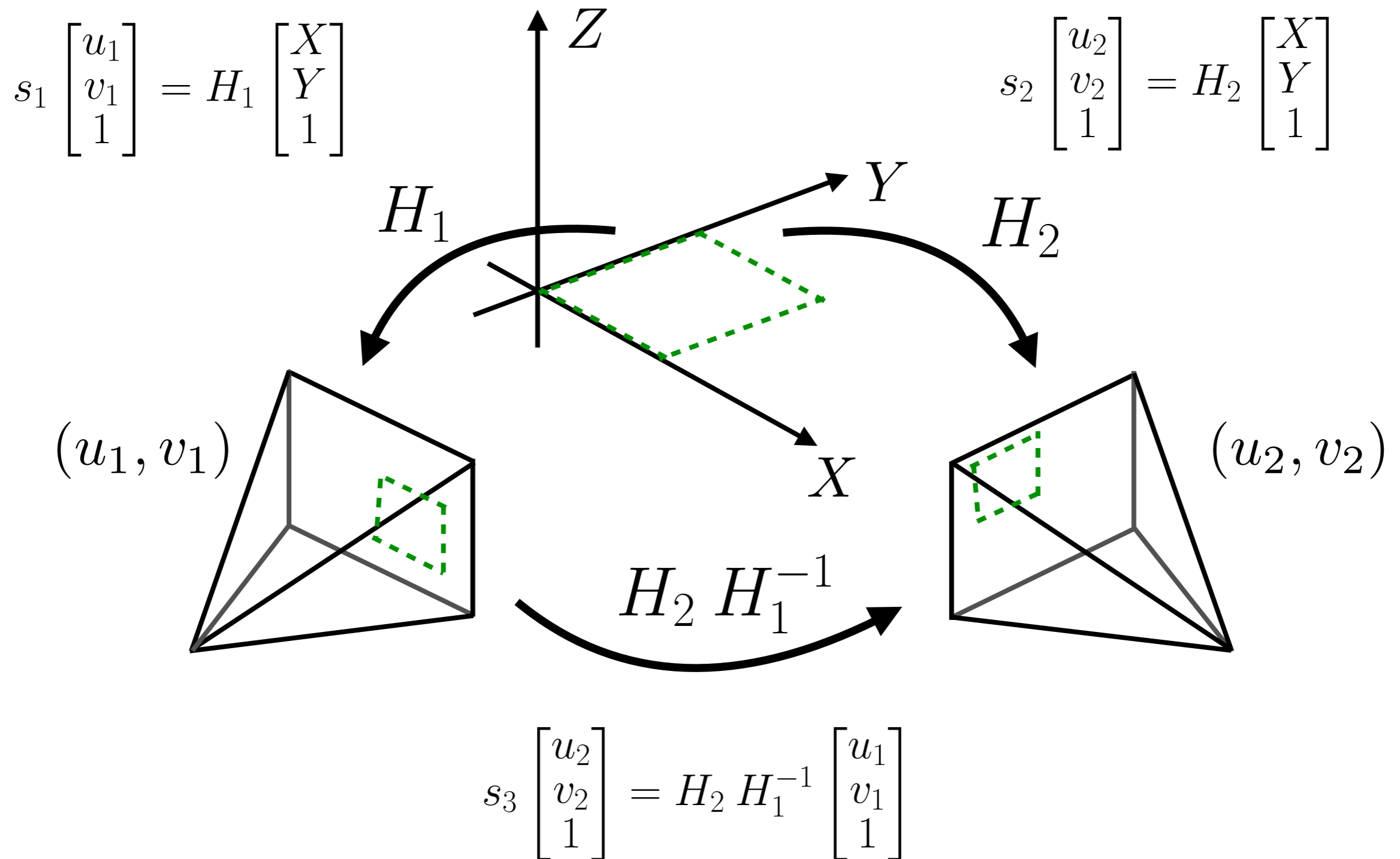
Projective Homography

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{14} \\ p_{21} & p_{22} & p_{24} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Linear (2d) Affine

# Viewing a Plane

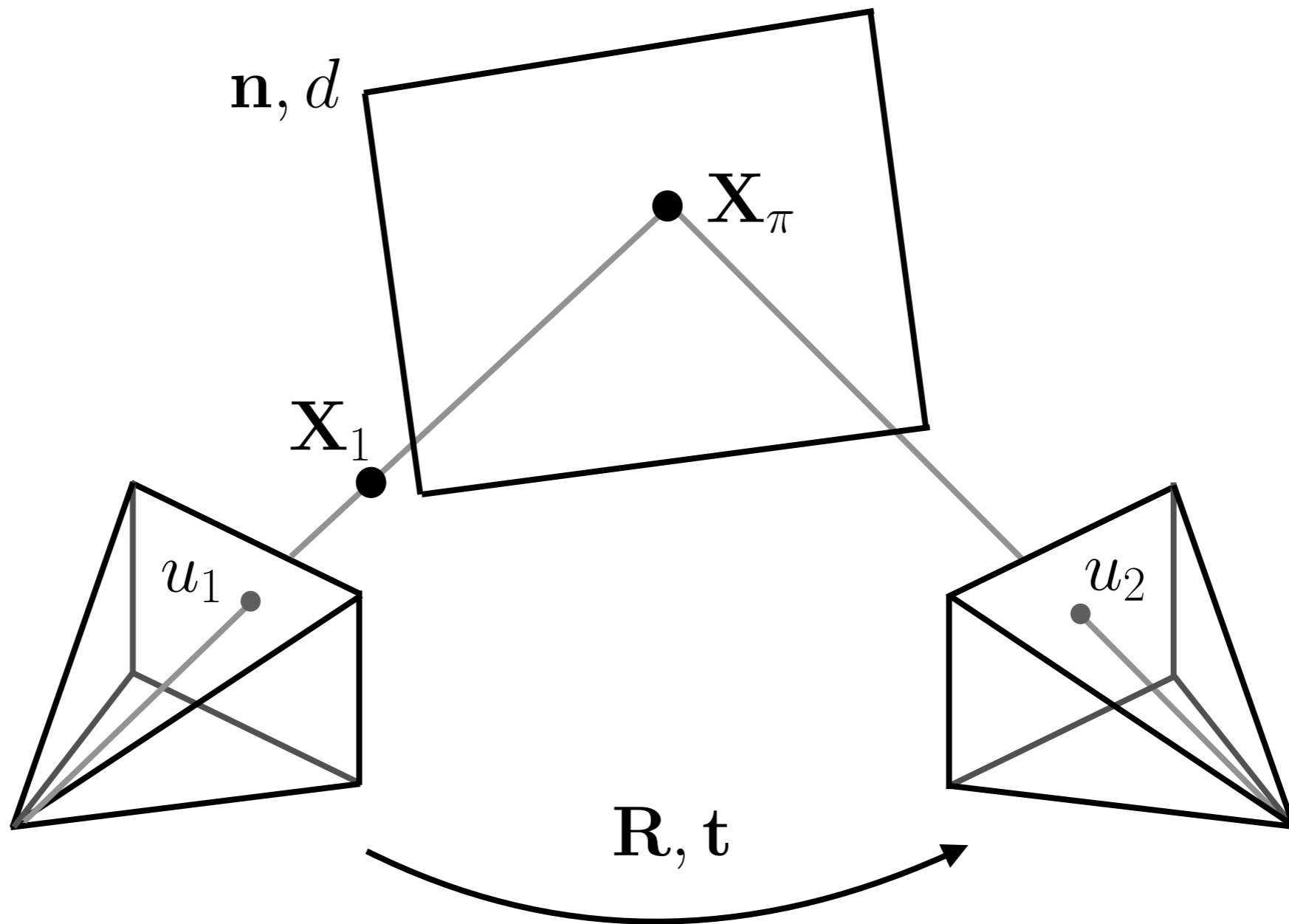
- Consider a pair of cameras viewing a plane





# Scene Plane

- What is the form of  $H$  in terms of scene parameters?



# Homography (between two views) induced by plane in 3D

assume  $C = [0, 0, 0, 1]$   
 $C \notin \Pi \Rightarrow \pi_4 \neq 0$ , e.g.  $\pi_4 = 1$   
 thus,  $\Pi = \begin{bmatrix} v \\ 1 \end{bmatrix}$ ,  $v^{3 \times 1}$

$X' = H \cdot X$  where  $H = H(P, P')$

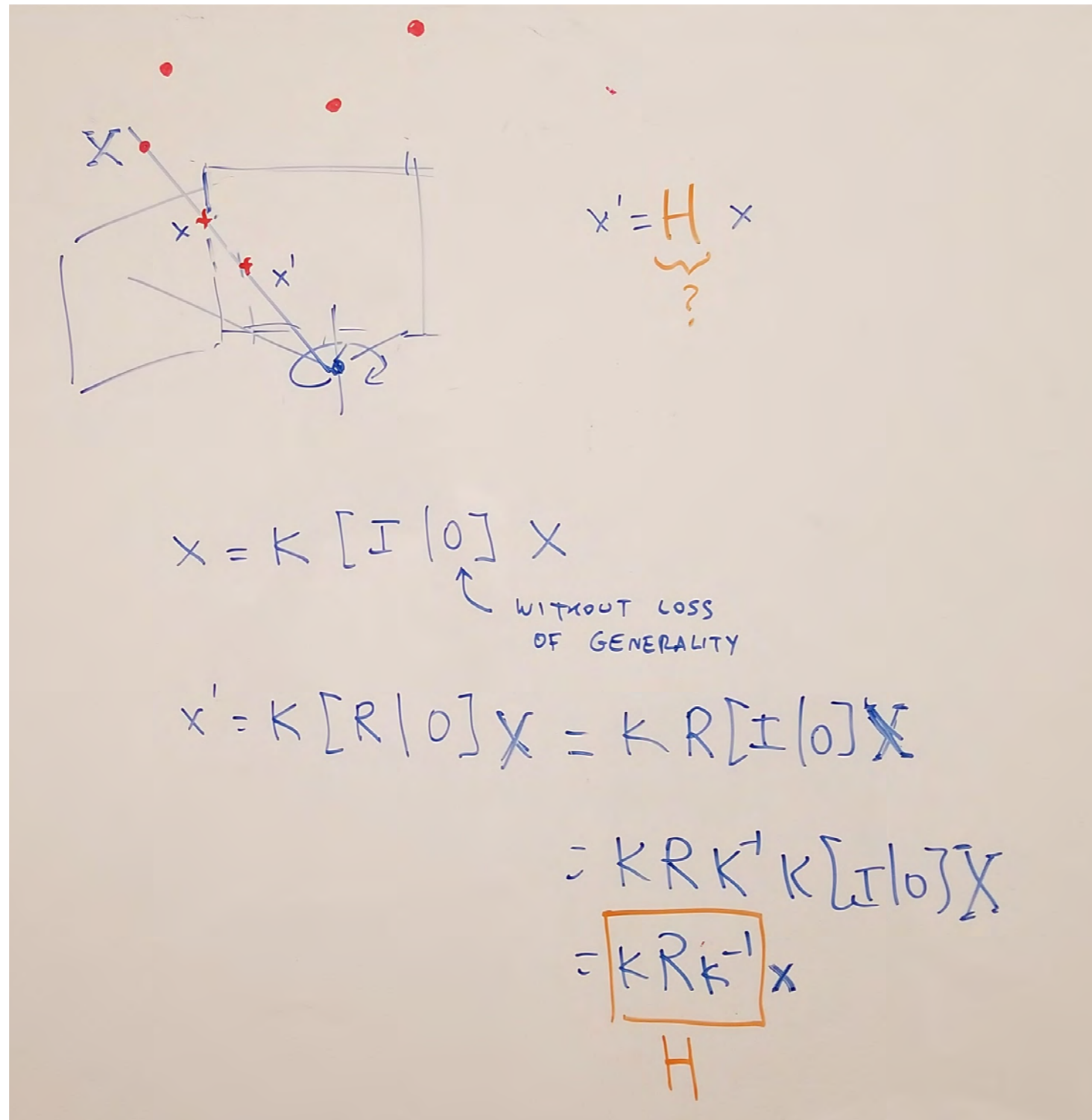
$P_E = K P$   $P'_E = K' P'$  ← adding calibration  
 $K' X' = K' H X = K' H K^{-1} K \cdot X$   
 $\downarrow$   
 $\underbrace{K' P'}_{P'_E} X_{\Pi} = \underbrace{K' H K^{-1}}_{H'} \underbrace{K P}_{P_E} X_{\Pi}$

$P = [I | 0]$   $P' = [A | a]$   
 $X = P X_{\Pi}$   $X' = P' X_{\Pi}$

ray:  $X = \begin{bmatrix} x \\ \rho \end{bmatrix}$  because  $P \cdot X = [I | 0] \begin{bmatrix} x \\ \rho \end{bmatrix} = X$   
 equation  $\forall \rho \in \mathbb{R}$

ON THE OTHER HAND,  $X \cap \Pi = X_{\Pi}$  i.e.  $\Pi^T \begin{bmatrix} x \\ \rho \end{bmatrix} = 0$   
 $\therefore [v^T, 1] \begin{bmatrix} x \\ \rho \end{bmatrix} = 0 \Rightarrow v^T x + \rho = 0 \Rightarrow \rho = -v^T x$   
 $\therefore X = \begin{bmatrix} x \\ -v^T x \end{bmatrix} \Rightarrow X' = P' X = [A | a] \begin{bmatrix} x \\ -v^T x \end{bmatrix} = Ax - av^T x$   
 $= (A - av^T) X$

# Homography (between two views) induced by camera rotation



The diagram illustrates the derivation of a homography matrix  $H$  for a camera rotation. It shows two camera centers (blue dots) and their respective image planes (blue rectangles). A point  $X$  in the world is projected onto the first image plane as  $x$  and onto the second image plane as  $x'$ . The rotation of the second camera is indicated by a curved arrow. The derivation of the homography matrix  $H$  is shown as follows:

$$x = K [I | 0] X$$

WITHOUT LOSS OF GENERALITY

$$x' = K [R | 0] X = K R [I | 0] X$$
$$= K R K^{-1} K [I | 0] X$$
$$= \boxed{K R K^{-1}} X$$

$H$

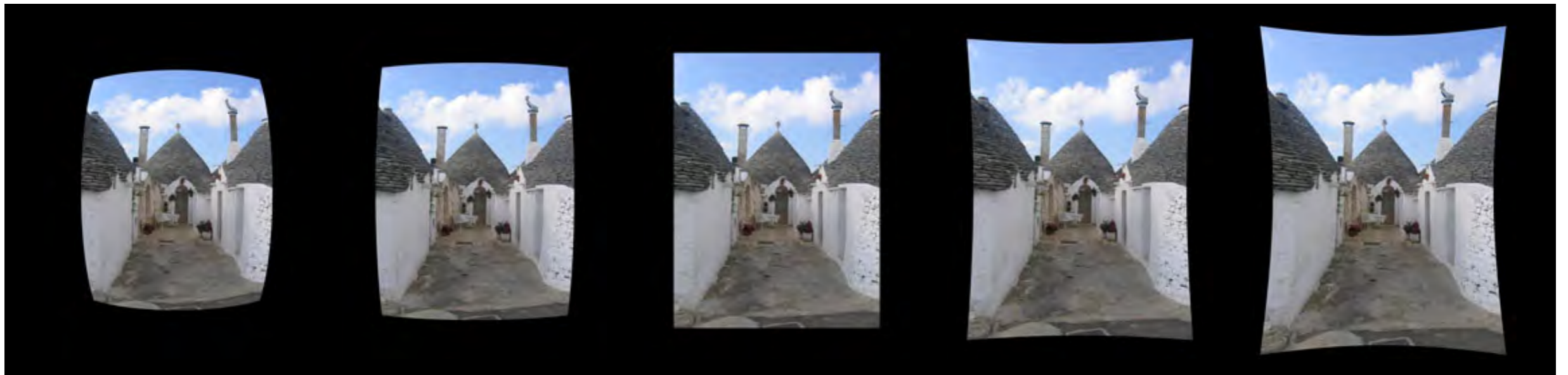
Handwritten equation:  $x' = H x$





# Radial Distortion

- In perspective (rectilinear) projection, straight lines in the world map to straight lines in the image, but many real imagers exhibit distortion towards the image edges



“barrel”

“pin cushion”

- A common first order model is  $\mathbf{x}' = (1 + \kappa|\mathbf{x}|^2)\mathbf{x}$
- Wide-angle imagers may have very different projection models, e.g., for equidistant fisheye  $r \propto \theta$

# Linear/Affine

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



viewing plane

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$



# Projective

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}$$



$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

(Homography)



# Next Lecture

- RANSAC