An Alternative Formulation of Natural C^2-Interpolating Splines

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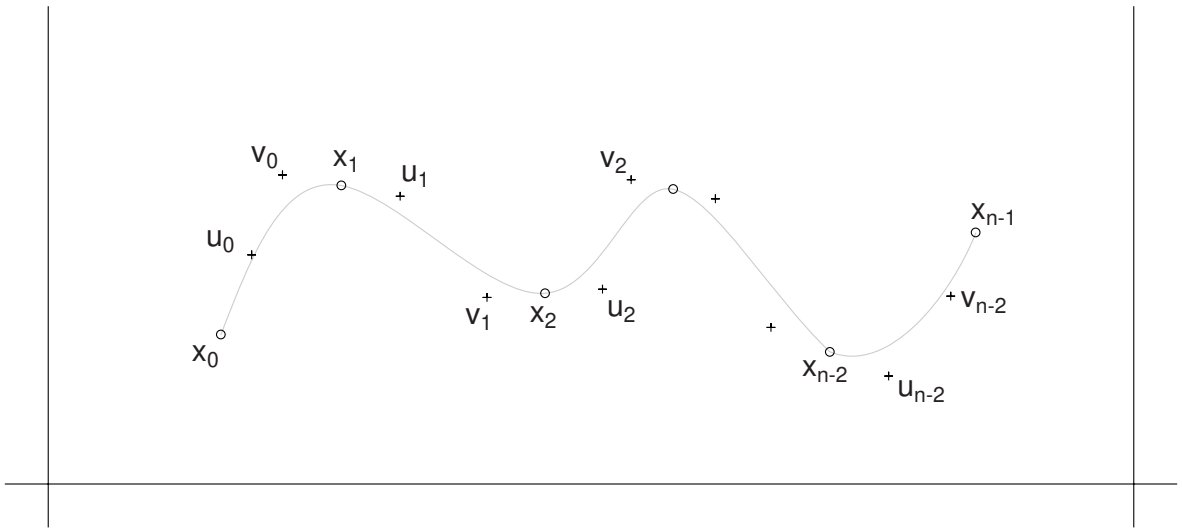
This document presents another (IMO easier) way to produce natural C^2-interpolating splines. I derived this by starting from the interactive method for constructing them given in lecture.

Here's a simple C^2-interpolating spline:



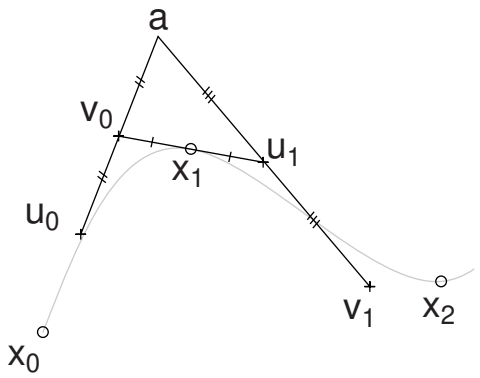
Note that all the variables in this document represent vector-valued quantities. You can do all this algebra componentwise if you choose to implement this method.

The interactive construction method supposes that the user has given all the x_is, as well as u_0 and v_0. From this it is trivial to construct the rest of the u_is and v_is needed to draw the spline using cubic Bézier segments.

For *natural* C^2-interpolating splines, though, we are not given u_0 and v_0. Instead, we are told that the second derivative at the endpoints of the spline (at x_0 and x_{n-1}) must be zero. This condition is enough to uniquely specify the curve. My approach will be to calculate what u_0

and v_0 must be to satisfy this condition. Once we have these points, we can draw the curve easily as before.

We first need to calculate u_{n-2} and v_{n-2} in terms of u_0 and v_0. This can be done by symbolically propagating the construction of of the u_is and v_is along the curve. Remember how given one u, v pair we construct the next pair with an A-frame:



$$\begin{aligned} u_1 &= x_1 + (x_1 - v_0) \\ &= -v_0 + 2x_1 \\ a &= v_0 + (v_0 - u_0) \\ &= 2v_0 - u_0 \end{aligned} \qquad \begin{aligned} v_1 &= u_1 + (u_1 - a) \\ &= u_1 + (u_1 - 2v_0 + u_0) \\ &= 2u_1 - 2v_0 + u_0 \\ &= 2(2x_1 - v_0) - 2v_0 + u_0 \\ &= u_0 - 4v_0 + 4x_1 \end{aligned}$$

Constructing the next A-frame gives us the next pair u, v:

$$\begin{aligned} u_2 &= -v_1 + 2x_2 \\ v_2 &= u_1 - 4v_1 + 4x_2 \end{aligned}$$

or

$$\begin{aligned} u_2 &= -(u_0 - 4v_0 + 4x_1) + 2x_2 \\ &= -u_0 + 4v_0 + (-4x_1 + 2x_2) \\ v_2 &= -v_0 + 2x_1 - 4(u_0 - 4v_0 + 4x_1) + 4x_2 \\ &= -4u_0 + 15v_0 + (-14x_1 + 4x_2) \end{aligned}$$

Note that each u, v pair is a linear combination of u_0 and v_0. Keep doing this propagation until you get to the last u, v pair, and let a through f represent the coefficients that come out of the propagation:

$$u_{n-2} = au_0 + bv_0 + c \quad (1)$$
$$v_{n-2} = du_0 + ev_0 + f \quad (2)$$

(Note that a is **not** the same as the a in the construction diagram above.) We have four unknowns (u_0, v_0, u_{n-2}, v_{n-2}), but so far, only two equations. To get a single solution, we need two more equations. Conveniently, the second-derivative-zero condition gives us one more equation at each end of the curve:

$$Q(u) = \begin{bmatrix} u^3 & u^2 & u^1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & \\ -3 & 3 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$
$$Q''(u) = \begin{bmatrix} 6u & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & \\ -3 & 3 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$
$$Q''(0) = 6V_0 - 12V_1 + 6V_2$$
$$Q''(1) = 6V_1 - 12V_2 + 6V_3$$

$$0 = 6x_0 - 12u_0 + 6v_0 \quad (3)$$
$$0 = 6u_{n-2} - 12v_{n-2} + 6x_{n-1} \quad (4)$$

Putting equations (1)–(4) together gives you a system of four equations and four unknowns. Some more tedious algebra gives you:

$$u_0 = \frac{x_0(2e - b) - 2f + c + x_{n-1}}{2d - a + 4e - 2b}$$
$$v_0 = 2u_0 - x_0$$

Voilà. The curve can now be drawn using the easy "interactive" method from lecture. A similar derivation works for the wrap-around version (replacing the two equations derived from the second-derivative condition with two derived from the wrapping condition).