

Parametric surfaces

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CSEP 557
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Reading

Optional reading:

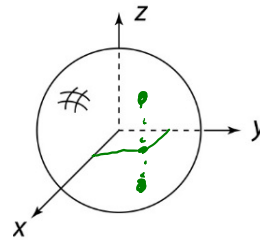
- ◆ Angel and Shreiner readings for “Parametric Curves” lecture, with emphasis on 10.1.2, 10.1.3, 10.1.5, 10.6.2, 10.7.3, 10.9.4.
- ◆ Marschner and Shirley, 2.5.

Further reading

- ◆ Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987.

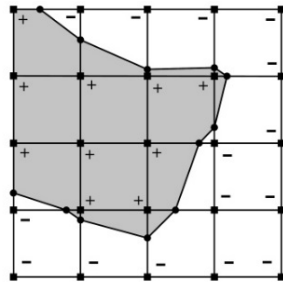
Mathematical surface representations

- ◆ Explicit $z = f(x, y)$ (a.k.a., a “height field”)
 - what if the curve isn’t a function, like a sphere?

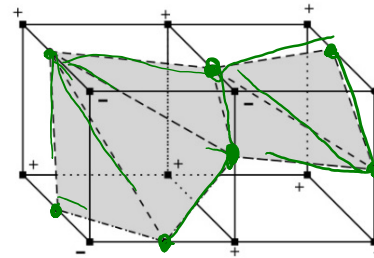


- ◆ Implicit $g(x, y, z) = 0$

$$g(x, y, z) = x^2 + y^2 + z^2 - r^2$$



Isocontour from “marching squares”



Isocontour from “marching cubes”

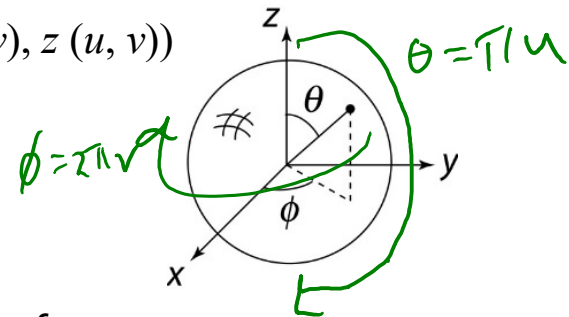
- ◆ Parametric $S(u, v) = (x(u, v), y(u, v), z(u, v))$

- For the sphere:

$$x(u, v) = r \cos(2\pi v) \sin(\pi u)$$

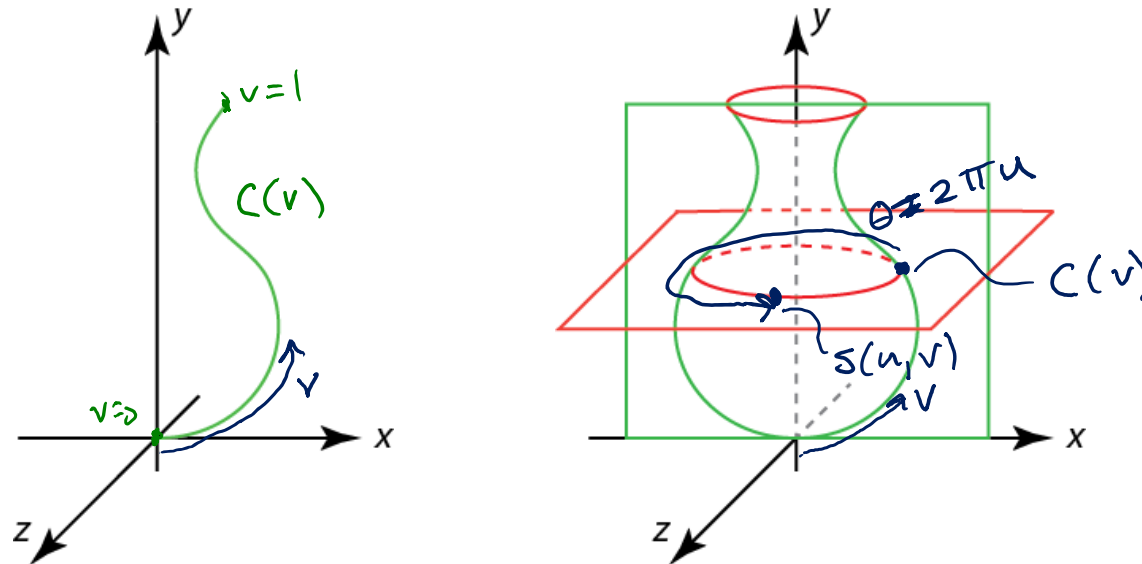
$$y(u, v) = r \sin(2\pi v) \sin(\pi u)$$

$$z(u, v) = r \cos(\pi u)$$



As with curves, we’ll focus on parametric surfaces.

Constructing surfaces of revolution



Given: A curve $C(v)$ in the xy -plane:

$$\underline{C(v)} = \begin{bmatrix} C_x(v) \\ C_y(v) \\ 0 \\ 1 \end{bmatrix}$$

Let $R_y(\theta)$ be a rotation about the y -axis.

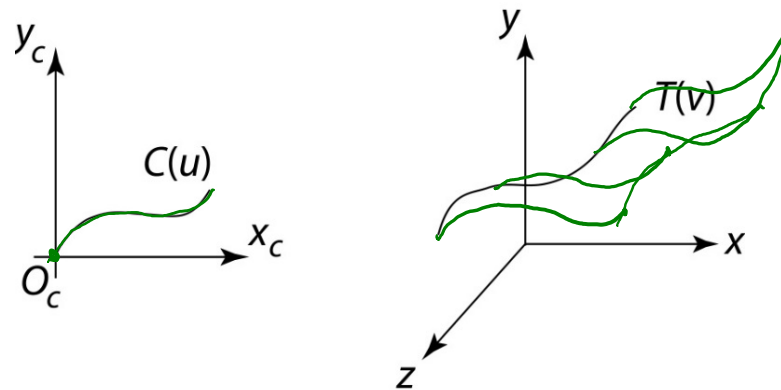
Find: A surface $S(u,v)$ which is $C(v)$ rotated about the y -axis, where $u, v \in [0, 1]$.

Solution: $S(u,v) = R_y(2\pi u) C(v)$

General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u, v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



More specifically:

- ◆ Suppose that $C(u)$ lies in an (x_c, y_c) coordinate system with origin O_c .
- ◆ For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

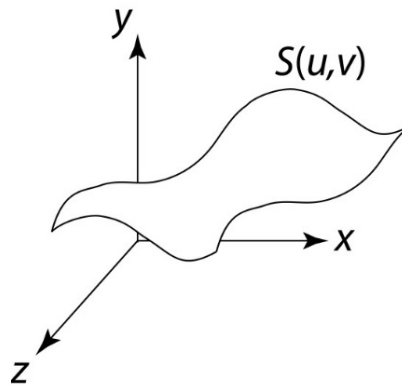
Orientation

The big issue:

- ◆ How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

1. **Fixed** (or **static**): Just translate O_c along $T(v)$.

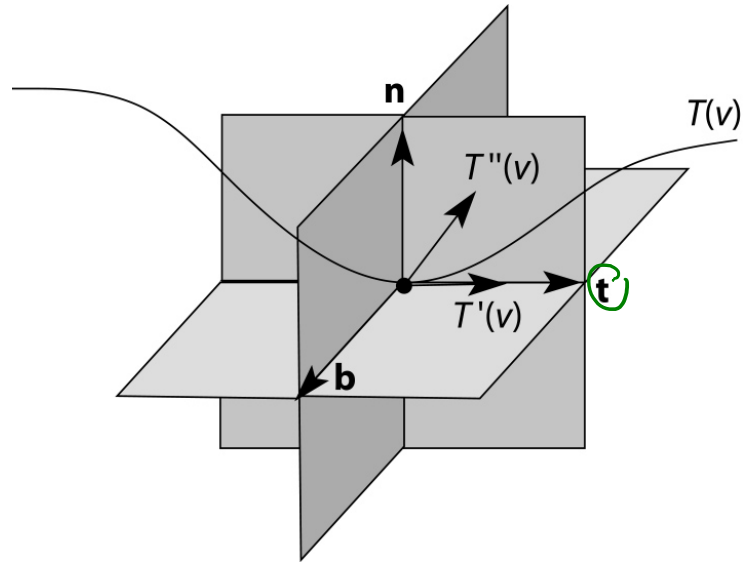


2. Moving. Use the **Frenet frame** of $T(v)$.

- ◆ Allows smoothly varying orientation.
- ◆ Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

$$\text{Tangent: } \mathbf{t}(v) = \text{normalize}[T'(v)]$$

$$\text{Binormal: } \mathbf{b}(v) = \text{normalize}[T'(v) \times T''(v)]$$

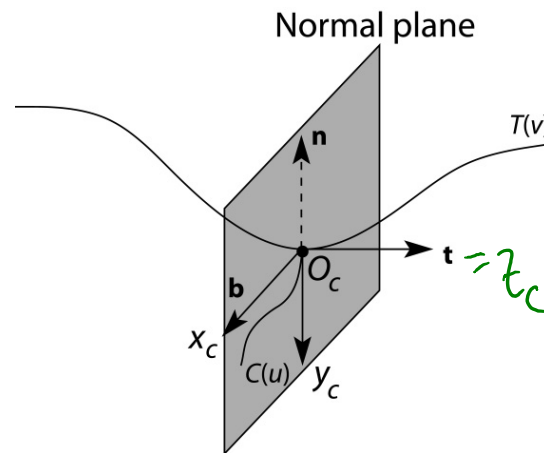
$$\text{Normal: } \mathbf{n}(v) = \mathbf{b}(v) \times \mathbf{t}(v)$$

As we move along $T(v)$, the Frenet frame $(\mathbf{t}, \mathbf{b}, \mathbf{n})$ varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

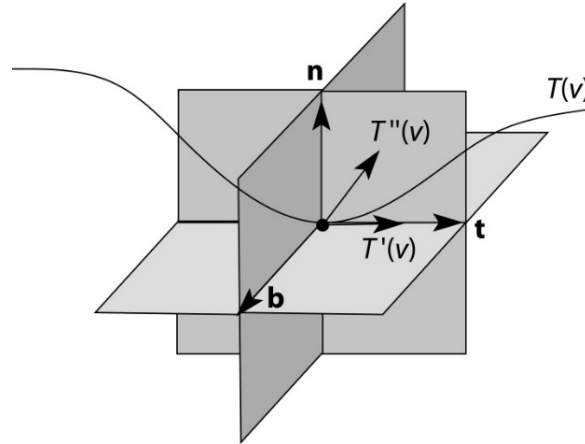
- ◆ Put $C(u)$ in the **normal plane** .
- ◆ Place O_c on $T(v)$.
- ◆ Align x_c for $C(u)$ with \mathbf{b} .
- ◆ Align y_c for $C(u)$ with $-\mathbf{n}$.



If $T(v)$ is a circle, you get a surface of revolution exactly!

Degenerate frames

Let's look back at where we computed the coordinate frames from curve derivatives:



$$\begin{aligned} \rightarrow t &= \text{norm}(T'(v)) \\ b &= \text{norm}(T'(v) \times T''(v)) \\ n &= b \times t \end{aligned}$$

Where might these frames be ambiguous or undetermined?

$$T'(v) = 0 \Rightarrow \text{arc length parameterization } T'(s)$$

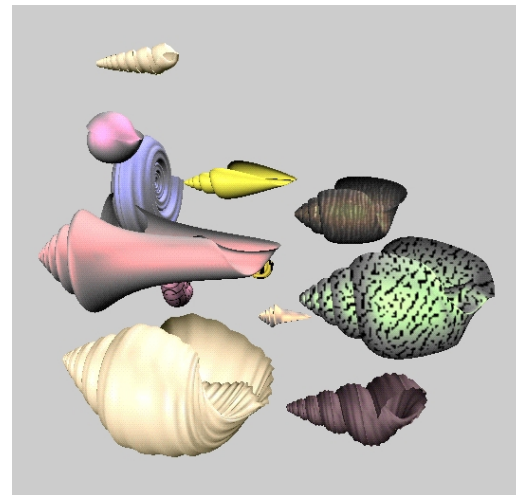
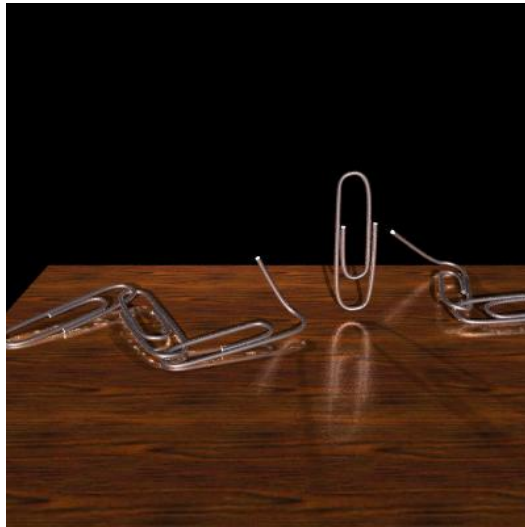
$$T''(v) = 0 \rightarrow T''(s) = 0$$



Variations

Several variations are possible:

- ◆ Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- ◆ Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- ◆ ...



Tensor product Bézier surfaces

Bézier control polygon

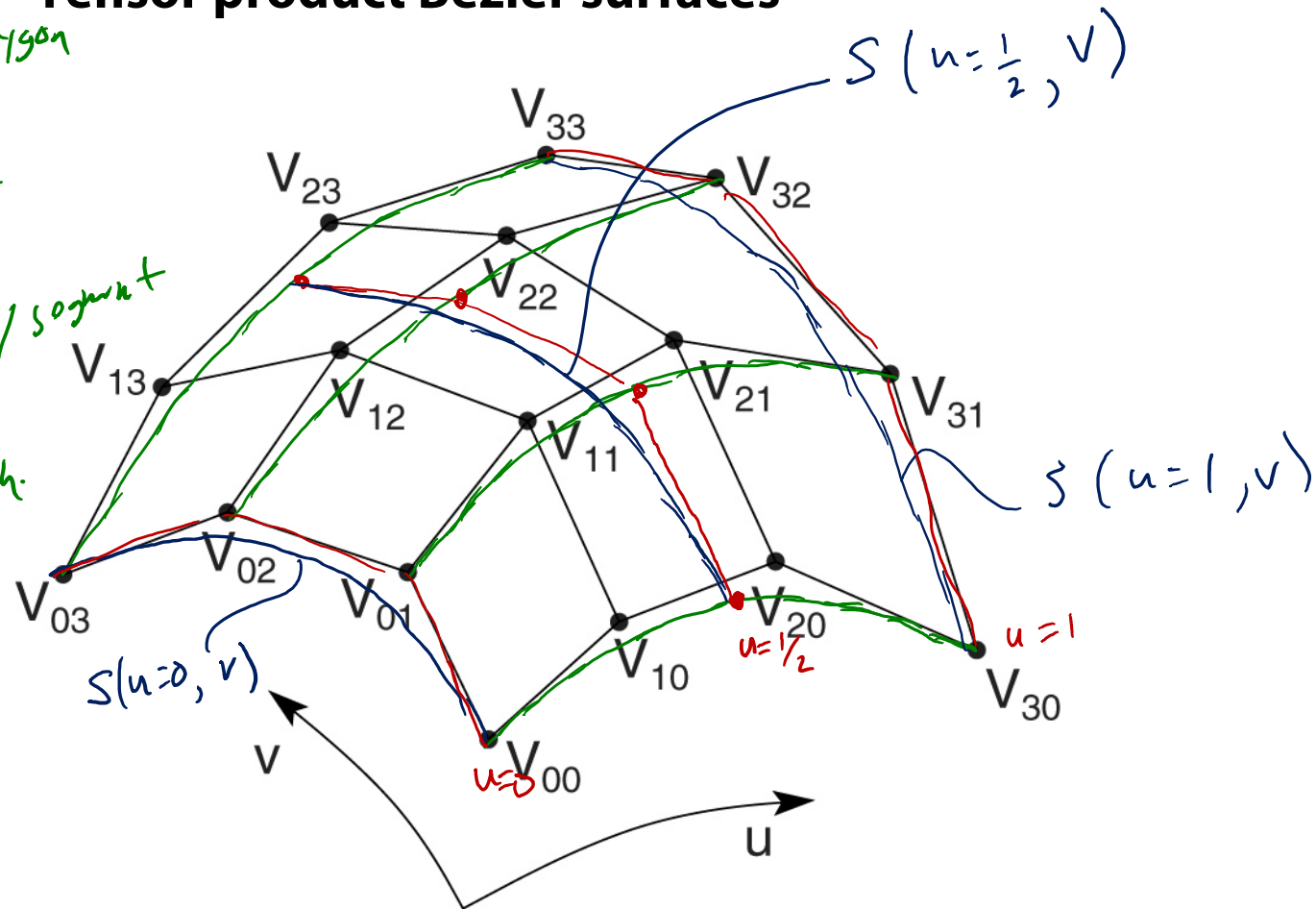


" " net

Cubic bézier curve/segment



" " patch

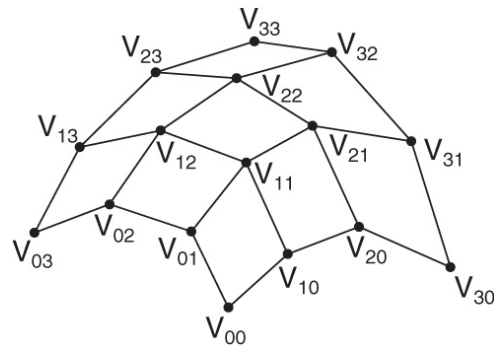


Given a grid of control points V_{ij} , forming a **control net**, construct a surface $S(u, v)$ by:

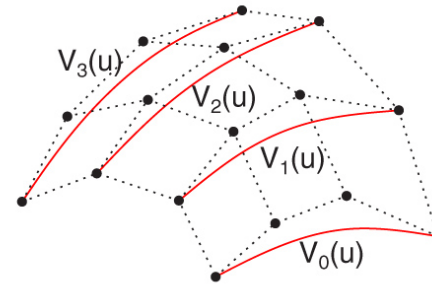
- ♦ treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0(u), \dots, V_n(u)$.
- ♦ treating $V_0(u), \dots, V_n(u)$ as control points for a curve parameterized by v .

Tensor product Bézier surfaces, cont.

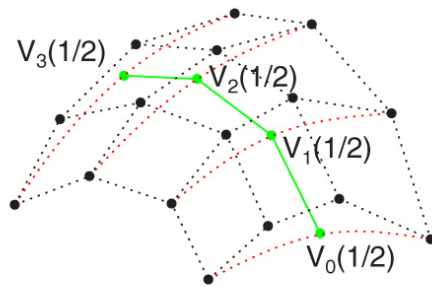
Let's walk through the steps:



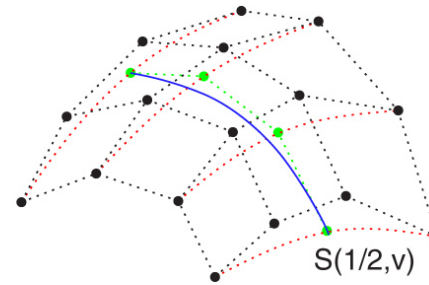
Control net



Control curves in u



Control polygon at $u=1/2$



Curve at $S(1/2, v)$

Which control points are always interpolated by the surface?

4 corners

Polynomial form of Bézier surfaces

Recall that cubic Bézier *curves* can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^n V_i b_i(u) \quad \leftarrow$$

A tensor product Bézier surface can be written as:

$$S(u, v) = \sum_{i=0}^n \sum_{j=0}^n V_{ij} b_i(u) b_j(v)$$

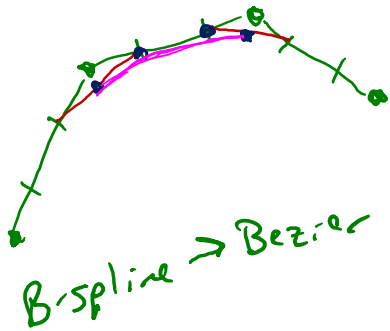
In the previous slide, we constructed curves along u , and then along v . This corresponds to re-grouping the terms like so:

$$S(u, v) = \sum_{j=0}^n \left(\sum_{i=0}^n V_{ij} b_i(u) \right) b_j(v)$$

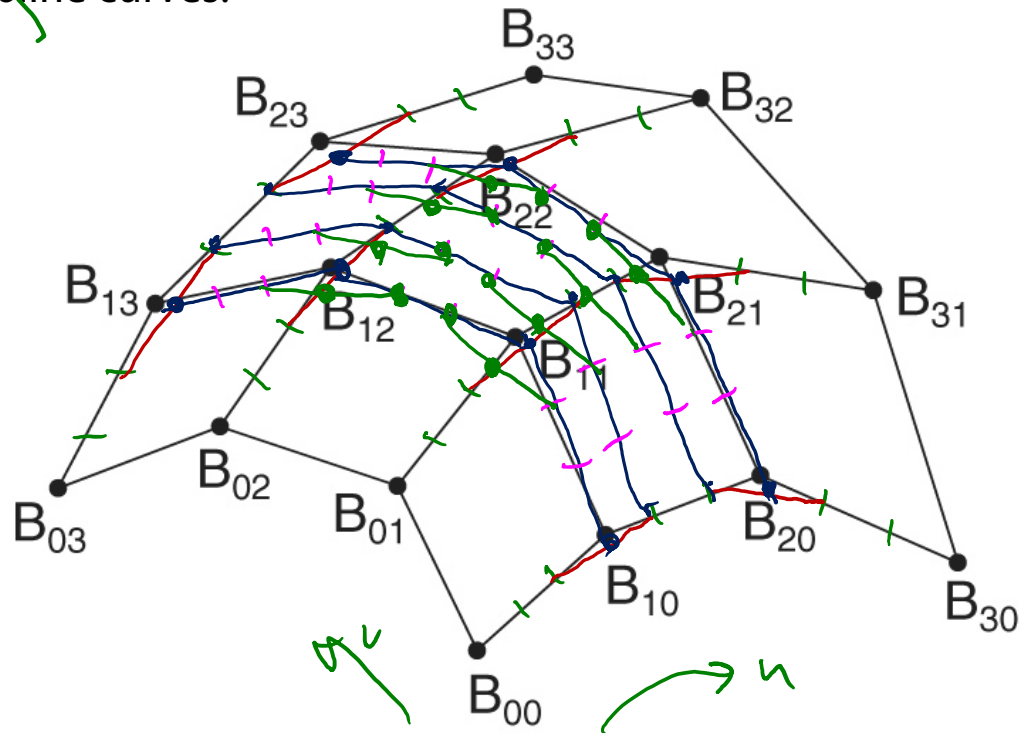
But, we could have constructed them along v , then u :

$$S(u, v) = \sum_{i=0}^n \left(\sum_{j=0}^n V_{ij} b_j(v) \right) b_i(u)$$

Tensor product B-spline surfaces

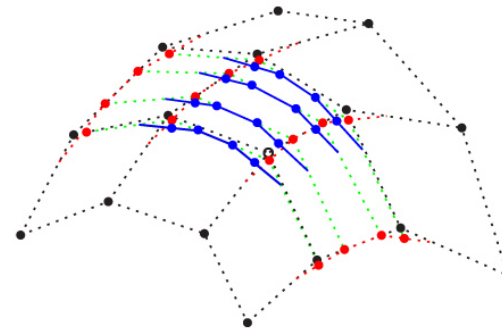
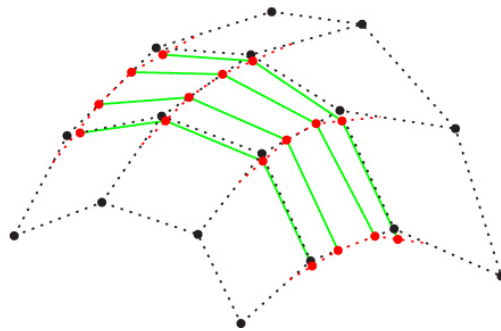
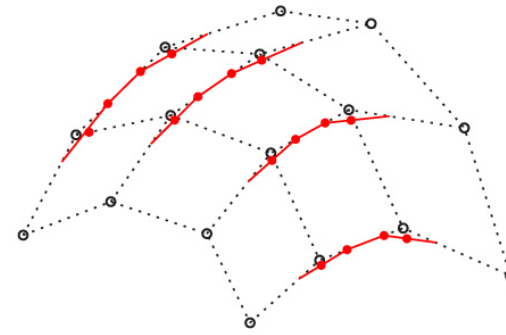
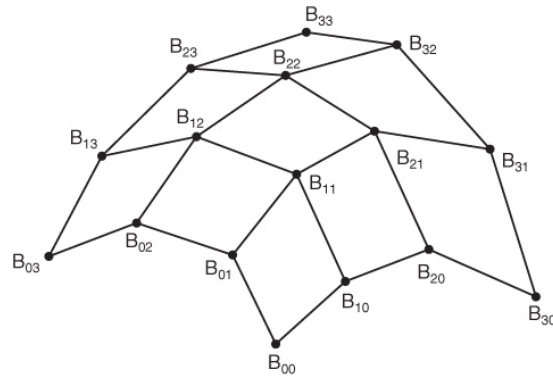


As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- ◆ treat rows of B as control points to generate Bézier control points in u .
- ◆ treat Bézier control points in u as B-spline control points in v .
- ◆ treat B-spline control points in v to generate Bézier control points in u .

Tensor product B-spline surfaces, cont.

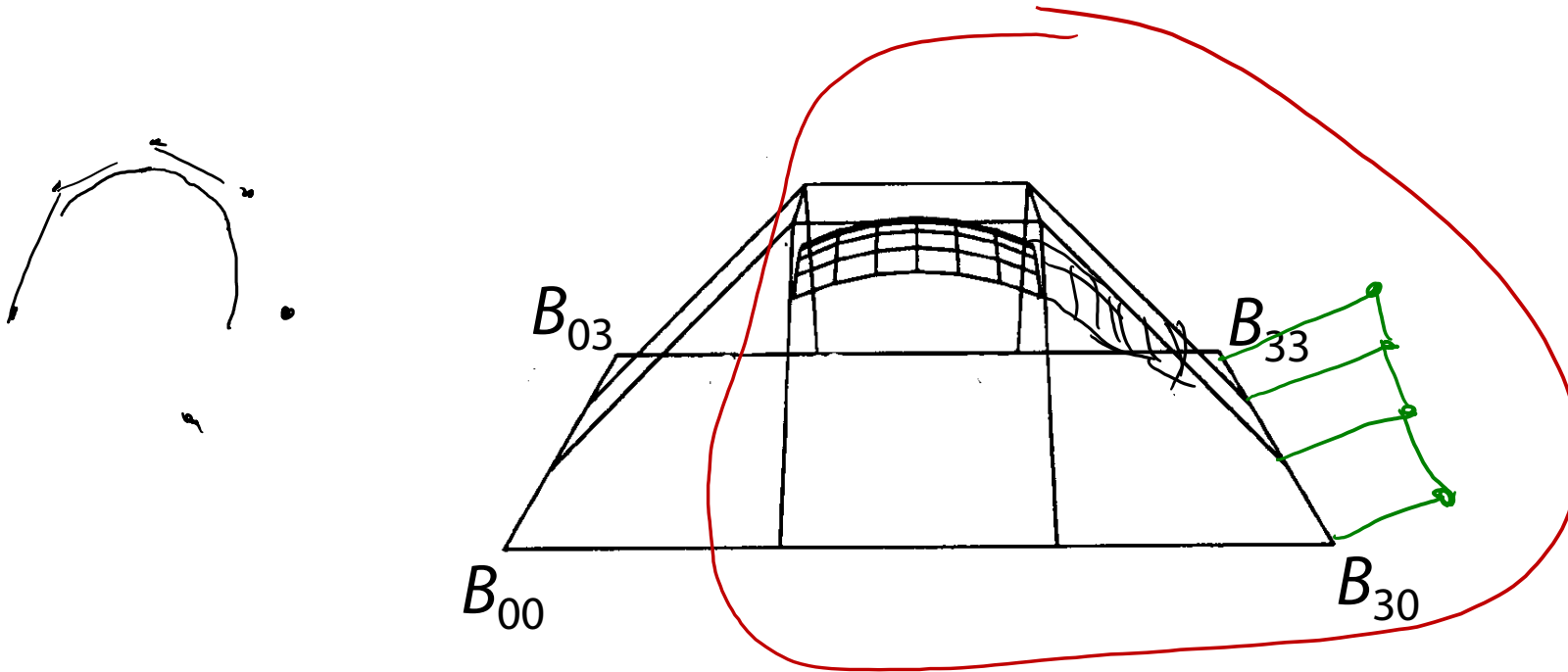


Which B-spline control points are always interpolated by the surface?

None

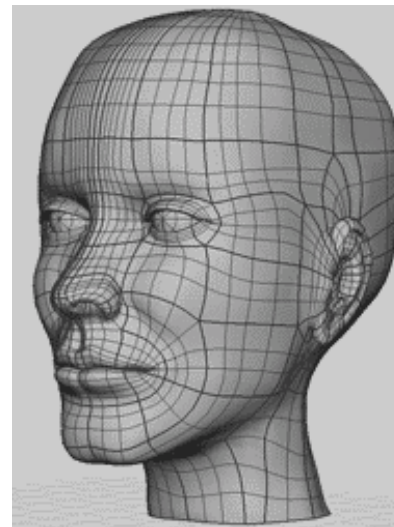
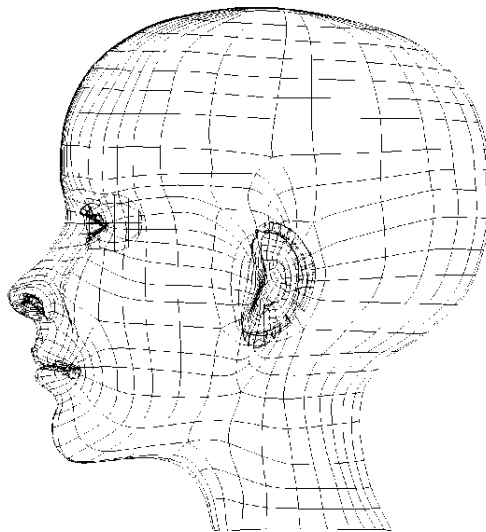
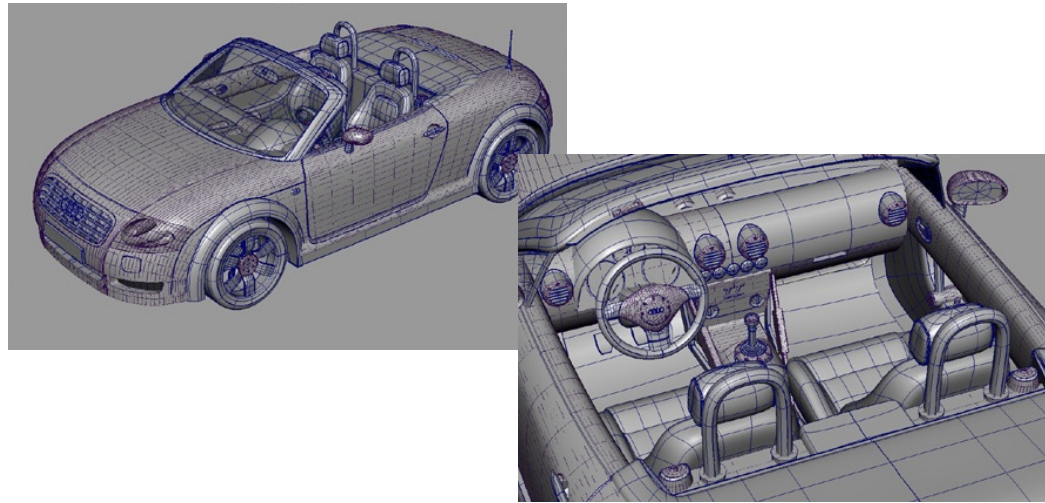
Tensor product B-splines, cont.

Another example:



NURBS surfaces

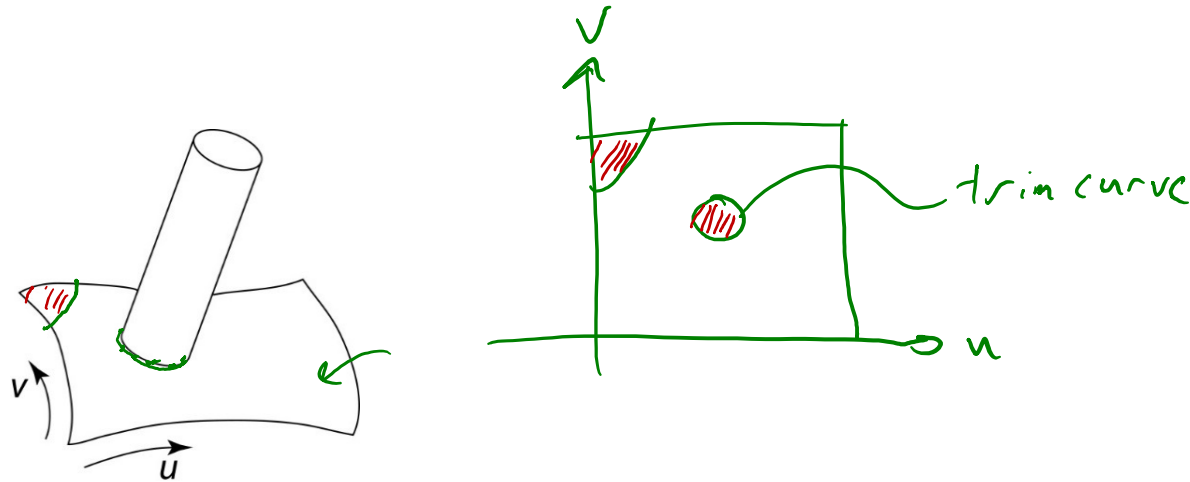
Uniform B-spline surfaces are a special case of NURBS surfaces.



Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:



We can do this by **trimming** the u - v domain.

- ◆ Define a closed curve in the u - v domain (a **trim curve**)
- ◆ Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

Summary

What to take home:

- ◆ How to construct swept surfaces from a profile and trajectory curve:
 - with a fixed frame
 - with a Frenet frame
- ◆ How to construct tensor product Bézier surfaces
- ◆ How to construct tensor product B-spline surfaces