Affine transformations

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CSEP 557
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Reading

Optional reading:

- Angel and Shreiner: 3.1, 3.7-3.11
- Marschner and Shirley: 2.3, 2.4.1-2.4.4, 6.1.1-6.1.4, 6.2.1, 6.3

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

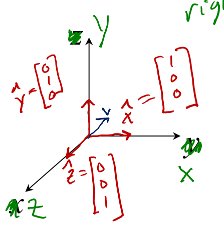
Vector representation

We can represent a **point**, $\mathbf{p} = (x, y)$, in the plane or $\mathbf{p} = (x, y, z)$ in 3D space:

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- as row vectors $\begin{bmatrix} x & y \end{bmatrix}$ $\begin{bmatrix} x & y & z \end{bmatrix}$

Canonical axes

right handed coord. system

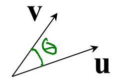


$$v = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$
$$= v_x\begin{bmatrix}1\\0\\0\end{bmatrix} + v_y\begin{bmatrix}0\\1\\0\end{bmatrix} + v_z\begin{bmatrix}0\\0\\1\end{bmatrix}$$
$$= \begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}$$

$\hat{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\hat{y} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\hat{z} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

right hand rule

Vector length and dot products



$v = \begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}$ $\quad u = \begin{bmatrix}u_x\\u_y\\u_z\end{bmatrix}$

$$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$
$$u \cdot v = u_x v_x + u_y v_y + u_z v_z$$
$$u \cdot v \stackrel{?}{=} v \cdot u \checkmark$$
$$u \cdot v = \begin{bmatrix}u_x & u_y & u_z\end{bmatrix}\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix} = u^T v = v^T u$$

$$v \cdot v = \|v\|^2$$

$\hat{u} = \frac{u}{\|u\|}$ $\quad \hat{v} = \frac{v}{\|v\|}$

$$u \cdot v = \|u\|\|v\| \cos\theta$$

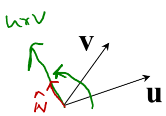
$\|\hat{u}\| = 1 = \|\hat{v}\|$

$u \cdot v = 0 \Rightarrow u \perp v$ (orthogonal)

$\|u\|, \|v\| \neq 0$

$\hat{u} \cdot \hat{v} = \cos\theta$

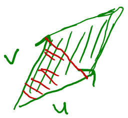
Vector cross products



$$u \times v = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix} = (u_y v_z - u_z v_y)\hat{x} + (u_z v_x - u_x v_z)\hat{y} + (u_x v_y - u_y v_x)\hat{z}$$

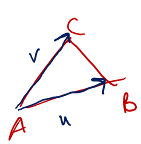
$$u \times v \,(\text{in } x\text{-}y\text{ plane}) = \begin{bmatrix}0\\0\\u_x v_y - u_y v_x\end{bmatrix}$$

$$u \times v = -v \times u$$
$$(u \times v) \cdot u = 0$$
$$(u \times v) \cdot v = 0$$
$$u \times v = \|u\|\|v\| \sin\theta \, \hat{N}$$



$$\text{Area}(\square_{u,v}) = \|u \times v\| = \|u\|\|v\| |\sin\theta|$$

$$\text{Area}(\Delta_{u,v}) = \frac{1}{2}\|u \times v\|$$



$$u = \alpha v \Rightarrow u \times v = 0$$

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}$$

If **p** is a column vector, M goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}ax+by\\cx+dy\end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix}x' & y'\end{bmatrix} = \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}a & c\\b & d\end{bmatrix} = \begin{bmatrix}ax+by & cx+dy\end{bmatrix}$$

We will use **column vectors**.

$$(AB)^T = B^T A^T$$
$$(AB)^{-1}(AB) = I$$
$$(AB)^{-1} A B = I$$
$$(AB)^{-1} A = B^{-1}$$
$$(AB)^{-1} = B^{-1} A^{-1}$$

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix M:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements $a, b, c, d\ldots$

Identity

Suppose we choose $a = d = 1, b = c = 0$:

- Gives the **identity** matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Doesn't move the points at all

$$x' = x$$
$$y' = y$$

Scaling

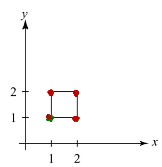
Suppose we set $b = c = 0$, but let a and d take on any *positive* value:

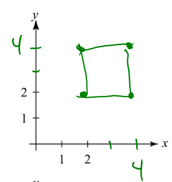
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

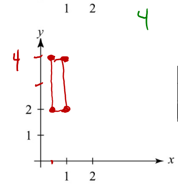
- Provides **differential (non-uniform) scaling** in x and y:

$$x' = ax \leftarrow$$
$$y' = dy \leftarrow$$





$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

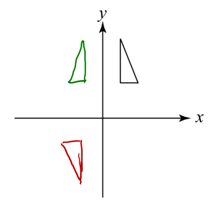
Reflection, mirror

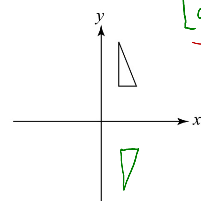
Suppose we keep $b = c = 0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$





Shear

Now let's leave $a = d = 1$ and experiment with $b \ldots$

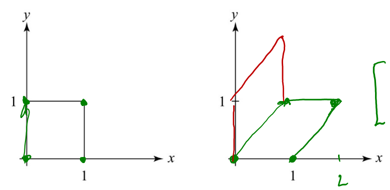
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$



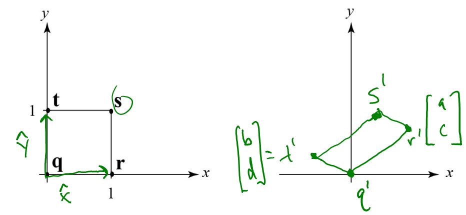
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \mathbf{q} & \mathbf{r} & \mathbf{s} & \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{q'} & \mathbf{r'} & \mathbf{s'} & \mathbf{t'} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$



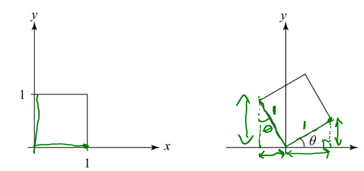
Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners $(1,0)$ and $(0,1)$ are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Translation

Homogeneous coordinates

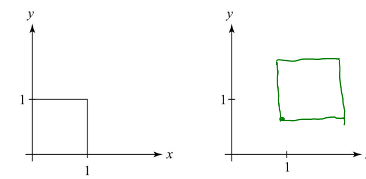
Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "w" component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + t_x \\ cx + dy + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + t \\ 1 \end{bmatrix}$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added w-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

Rotation about arbitrary points

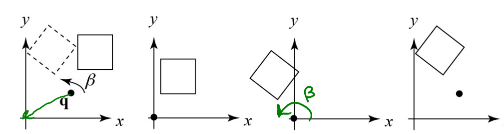
Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation by β, about any point $\mathbf{q} = [q_x \; q_y]^T$ with a matrix.

Let's do this with rotation and translation matrices of the form $R(\theta)$ and $T(\mathbf{t})$, respectively.

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(\mathbf{t}) = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$



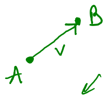
$M \neq$ $\quad T(-q)$ $\quad R(\beta)$ $\quad T(q)$

$$M = T(q)R(\beta)T(-q)$$

1. Translate $\mathbf{q}$ to origin
2. Rotate
3. Translate back

Points and vectors

Vectors have an additional coordinate of $w = 0$. Thus, a change of origin has no effect on vectors.



$A + v = B$

$v = B - A$

$= \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} - \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix}$

$= \begin{bmatrix} B_x - A_x \\ B_y - A_y \\ 0 \end{bmatrix}$

Q: What happens if we multiply a vector by an affine matrix?

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} a v_x + b v_y \\ c v_x + d v_y \\ 0 \end{bmatrix}$$

These representations reflect some of the rules of affine operations on points and vectors:

vector + vector → vector

scalar · vector → vector

point - point → vector

point + vector → point

point + point → chaos

scalar · vector + scalar · vector → vector

scalar · point + scalar · point → it depends

point + scalar · vector → point

$\alpha \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} + \beta \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \alpha A_x + \beta B_x \\ \alpha A_y + \beta B_y \\ \alpha + \beta \end{bmatrix}$

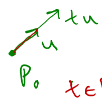
$\alpha + \beta = 1 \Rightarrow$ point

$\alpha + \beta = 0 \Rightarrow$ vector

else $\Rightarrow$ chaos

One useful combination of affine operations is:

$$P(t) = P_o + t\mathbf{u}$$



$t \in (-\infty, \infty)$ line

$t \in [0, \infty)$ ray

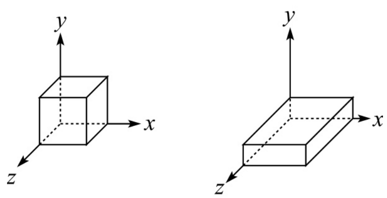
Q: What does this describe?

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

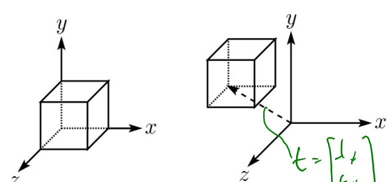
For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$t = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$

Rotation in 3D (cont'd)

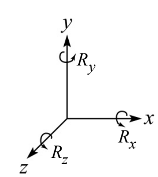
$R = \begin{bmatrix} u & v & w \end{bmatrix}$

These are the rotations about the canonical axes:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

$u \cdot u = 1$ $\quad u \cdot v = 0$

$v \cdot v = 1$ $\quad u \cdot w = 0$

$w \cdot w = 1$ $\quad v \cdot w = 0$

$R^T = R^{-1}$

$R^T R = \begin{bmatrix} - u^T - \\ - v^T - \\ - w^T - \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix}$

$= \begin{bmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{bmatrix} = I$

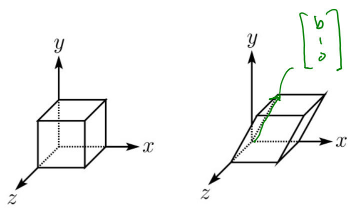
A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

quaternion $\begin{bmatrix} q_x \\ q_y \\ q_z \\ q_w \end{bmatrix}$

Shearing in 3D

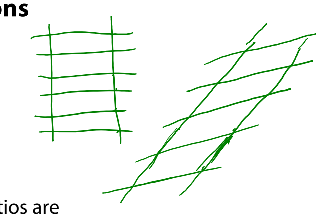
Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



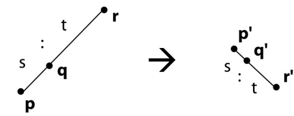
We call this a shear with respect to the x-z plane.

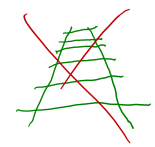
Properties of affine transformations



Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)





$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.