Affine transformations

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CSEP 557
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Reading

Required:

- Angel 3.1, 3.7-3.11

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

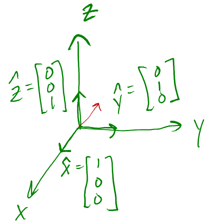
Vector representation

We can represent a **point**, $\mathbf{p} = (x, y)$, in the plane or $\mathbf{p} = (x, y, z)$ in 3D space

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- ~~as row vectors~~ $\cancel{\begin{bmatrix} x & y \end{bmatrix}}$ $\cancel{\begin{bmatrix} x & y & z \end{bmatrix}}$

Canonical axes

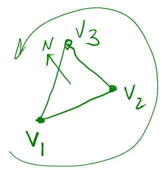


$\hat{z} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\hat{y} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\hat{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

right-handed coord. system

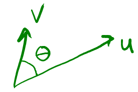
$V = \begin{bmatrix}V_x\\V_y\\V_z\end{bmatrix} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$

$\hat{x} \times \hat{y} = \hat{z}$



right-hand rule for Δ's

Vector length and dot products



$v = \begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}$ $u = \begin{bmatrix}u_x\\u_y\\u_z\end{bmatrix}$

$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

$u \cdot v = u_x v_x + u_y v_y + u_z v_z$

$v \cdot v = \|v\|^2$

$u \cdot v \stackrel{?}{=} v \cdot u$ ✓ yes

$u \cdot v = u^T v = v^T u = v \cdot u$

$u \cdot v = \|u\| \, \|v\| \cos\theta$

$u \cdot v = 0 \iff u \perp v$ (orthogonal)

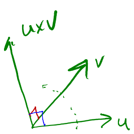
$(\|u\|, \|v\| \neq 0)$

Direction

$\hat{u} = \frac{u}{\|u\|}$

$\hat{u} \cdot \hat{v} = \cos\theta$

Vector cross products



$u \times v = \det \begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ u_x & u_y & u_z \\ v_x & v_y & v_z\end{bmatrix} = (u_y v_z - u_z v_y)\hat{x} + (u_z v_x - u_x v_z)\hat{y} + (u_x v_y - u_y v_x)\hat{z}$

$(u \times v) \cdot u = 0$

$(u \times v) \cdot v = 0$

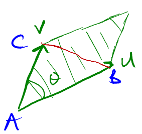
$u \times v = -v \times u$

$\|u \times v\| = \|u\| \, \|v\| \sin\theta = \text{Area}(\square_{u,v}) = 2\,\text{Area}(\Delta_{u,v})$

$\|\hat{u} \times \hat{v}\| = 1 \iff \hat{u} \perp \hat{v}$

$\|u \times v\| = 0 \iff u = \alpha v$

$(\|u\|, \|v\| \neq 0)$



$u = B - A$

$v = C - A$

$\text{Area}(\Delta_{ABC}) = \frac{1}{2}\|u \times v\|$

$N \sim u \times v$

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

If **p** is a column vector, M goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}ax + by\\ cx + dy\end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\cancel{\mathbf{p}' = \mathbf{p}M^T}$$

$$\cancel{\begin{bmatrix}x' & y'\end{bmatrix} = \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix}}$$

We will use **column vectors**.

$(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1}(AB) = I$

$(A^T)^{-1} = (A^{-1})^T = A^{-T}$

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix *M*:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d*…

Identity

Suppose we choose $a = d = 1$, $b = c = 0$:

- Gives the **identity** matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Doesn't move the points at all

Scaling

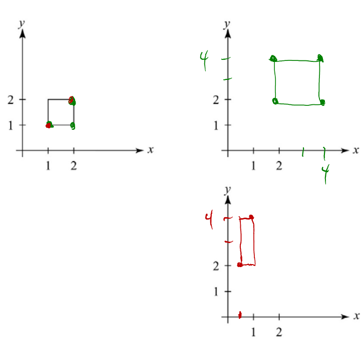
Suppose we set $b = c = 0$, but let *a* and *d* take on any *positive* value:

- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- Provides **differential (non-uniform) scaling** in *x* and *y*:

$$x' = ax$$
$$y' = dy$$



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$ uniform scaling

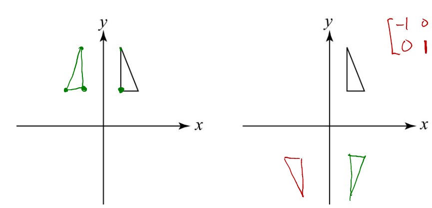
$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x \\ 2y \end{bmatrix}$$ non-uniform scaling

Mirror or Reflection

Suppose we keep $b = c = 0$, but let either *a* or *d* go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$R(180^\circ)$

Shear

Now let's leave $a = d = 1$ and experiment with b...

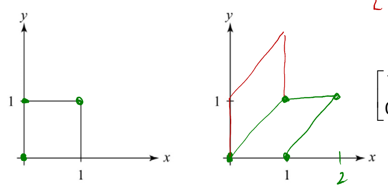
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

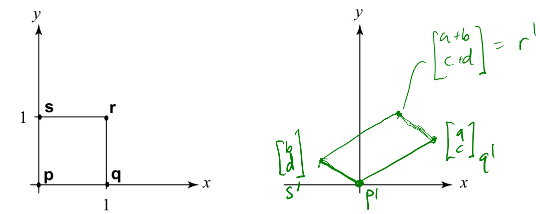
Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}[\mathbf{p} \quad \mathbf{q} \quad \mathbf{r} \quad \mathbf{s}] = [\mathbf{p'} \quad \mathbf{q'} \quad \mathbf{r'} \quad \mathbf{s'}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$

$$\begin{array}{cccc} & q' & & s' \end{array}$$



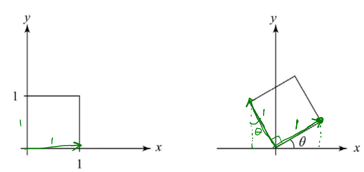
Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

translation

Homogeneous coordinates

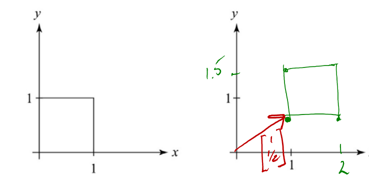
Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "*w*" component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D affine transformations always have a bottom row of [0 0 1].

$$\begin{bmatrix} ax + by + t_x \\ cx + dy + t_y \\ 1 \end{bmatrix}$$

An "affine point" is a "linear point" with an added *w*-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \\ 1 \end{bmatrix}$$

Rotation about arbitrary points

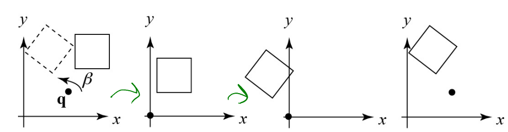
Until now, we have only considered rotation about the origin.

$$R(\beta) = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With homogeneous coordinates, you can specify a rotation by β, about any point $\mathbf{q} = [q_x \; q_y]^T$ with a matrix.

$$T(-q) = \begin{bmatrix} 1 & 0 & -q_x \\ 0 & 1 & -q_y \\ 0 & 0 & 1 \end{bmatrix}$$

Let's do this with rotation and translation matrices of the form R(θ) and T(**t**), respectively.



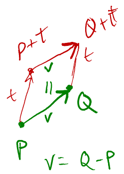
$$M \neq T(-q) \cdot R(\beta) \cdot T(q)$$

1. Translate **q** to origin
2. Rotate
3. Translate back

$$M = T(q)\,R(\beta)\,T(-q)$$

order of xforms is important

Points and vectors



$v = Q - P$

$= \begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix} - \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix}$

$= \begin{bmatrix} Q_x - P_x \\ Q_y - P_y \\ 0 \end{bmatrix}$

Vectors have an additional coordinate of $w = 0$. Thus, a change of origin has no effect on vectors.

Q: What happens if we multiply a vector by an affine matrix?

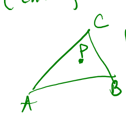
$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} a v_x + b v_y \\ c v_x + d v_y \\ 0 \end{bmatrix}$$

These representations reflect some of the rules of affine operations on points and vectors:

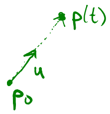
$$\alpha \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} + \beta \begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha P_x + \beta Q_x \\ \alpha P_y + \beta Q_y \\ \alpha + \beta \end{bmatrix}$$

$$\begin{cases} \text{point, if } \alpha + \beta = 1 \\ \text{vector, if } \alpha + \beta = 0 \\ \text{chaos, else} \end{cases}$$

vector + vector	→	vector
scalar · vector	→	vector
point - point	→	vector
point + vector	→	point
point + point	→	chaos
scalar · vector + scalar · vector	→	vector
scalar · point + scalar · point	→	it depends



$P = \alpha A + \beta B + \gamma C$



One useful combination of affine operations is:

$$\mathbf{p}(t) = \mathbf{p}_o + t\mathbf{u}$$

Q: What does this describe?

$t \in (-\infty, \infty) \Rightarrow$ line

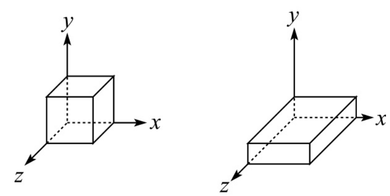
$t \in [0, \infty) \Rightarrow$ ray (half-line)

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

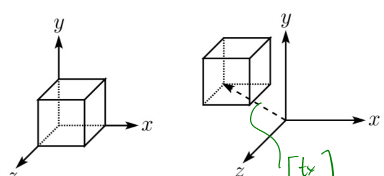
For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$\begin{bmatrix} t_x \\ t_y \\ t_z \\ 0 \end{bmatrix}$

Rotation in 3D (cont'd)

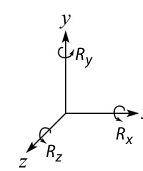
$R^{-T} = R$

These are the rotations about the canonical axes:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

$R = \begin{bmatrix} u & v & w \end{bmatrix}$

$u \cdot u = 1 \quad u \cdot v = 0$

$v \cdot v = 1 \quad u \cdot w = 0$

$w \cdot w = 1 \quad v \cdot w = 0$

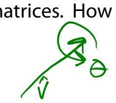
$R^T R = \begin{bmatrix} - & u^T & - \\ - & v^T & - \\ - & w^T & - \end{bmatrix}\begin{bmatrix} u & v & w \end{bmatrix}$

$= \begin{bmatrix} u^T u & u^T v & u^T w \\ v^T u & v^T v & v^T w \\ w^T u & w^T v & w^T w \end{bmatrix} = I$

$u^T v = u \cdot v$

$R^T R = I \quad R^T = R^{-1}$

A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

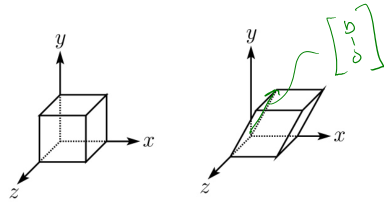


$q = \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_w \end{bmatrix}$

Shearing in 3D

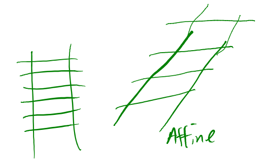
Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



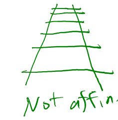
We call this a shear with respect to the x-z plane.

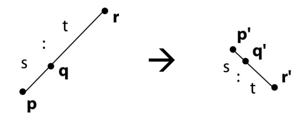
Properties of affine transformations



Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
  - (when transforming from N dimensions to N dimensions)
- Midpoints map to midpoints (in fact, ratios are always preserved)





$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Affine transformations in OpenGL

OpenGL maintains a "modelview" matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` **M** ← **I**
  - set **M** to identity
- `glTranslatef(t_x, t_y, t_z)` **M** ← **MT**
  - translate by (t_x, t_y, t_z)
- `glRotatef(θ, x, y, z)` **M** ← **MR**
  - rotate by angle θ about axis (x, y, z)
- `glScalef(s_x, s_y, s_z)` **M** ← **MS**
  - scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.