6. Affine transformations

Reading

Required:

- Watt, Section 1.1.

Further reading:

- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2^{nd} Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: *(x',y',z') =* ***f****(x,y,z).*

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

We'll start in 2D...

Representation

We can represent a **point**, **p** = (x,y), in the plane

- as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$
- as a row vector $\begin{bmatrix} x & y \end{bmatrix}$

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, M goes on the left:

$$\mathbf{p'} = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p'} = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix M:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d*…

Identity

Suppose we choose *a=d=1, b=c=0:*

- Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

Scaling

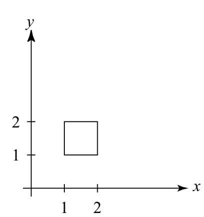
Suppose we set *b=c=0*, but let *a* and *d* take on any *positive* value:

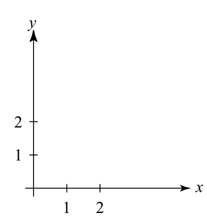
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

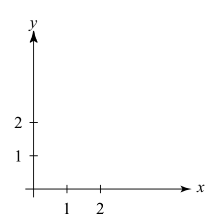
- Provides **differential scaling** in *x* and *y*:

$$x' = ax$$
$$y' = dy$$





$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

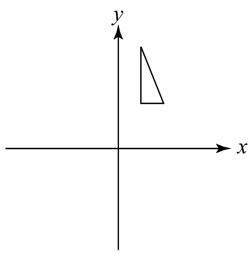


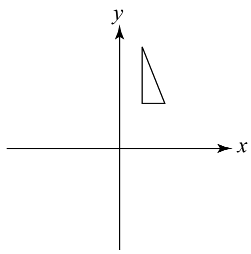
$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

Suppose we keep $b=c=0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$





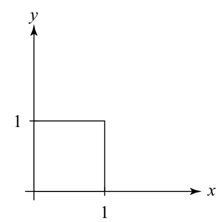
Now let's leave $a=d=1$ and experiment b. . . .

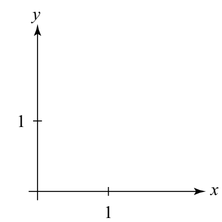
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$





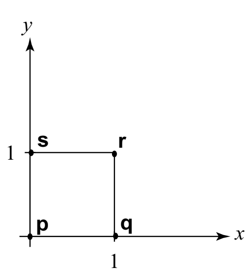
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

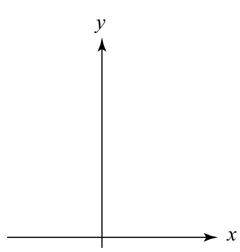
Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [\mathbf{p} \quad \mathbf{q} \quad \mathbf{r} \quad \mathbf{s}] = [\mathbf{p'} \quad \mathbf{q'} \quad \mathbf{r'} \quad \mathbf{s'}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$





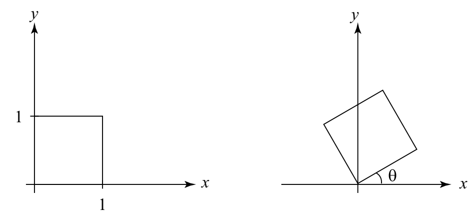
Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for “rotation about the origin”:



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$

Thus,

$$M = R(\theta) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

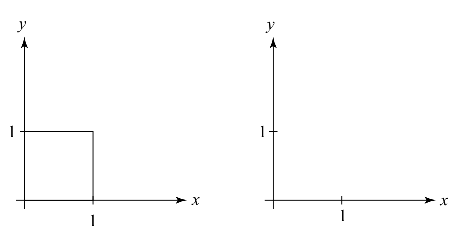
Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



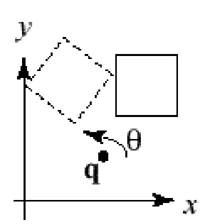
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, θ, about any point **q** = $[q_x \; q_y]^T$ with a matrix.



Q: how would you find the matrix for rotating about **q** by θ?

Rotation about arbitrary points

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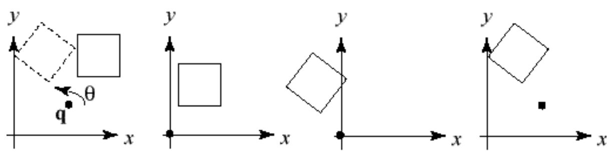


1. Translate **q** to origin
2. Rotate
3. Translate back

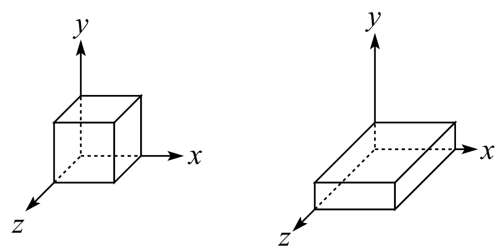
Note: Transformation order is important!!

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

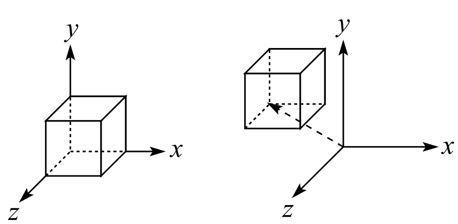
For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



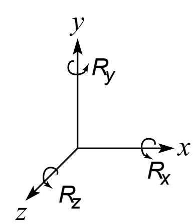
Rotation in 3D

Rotation now has more possibilities in 3D:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

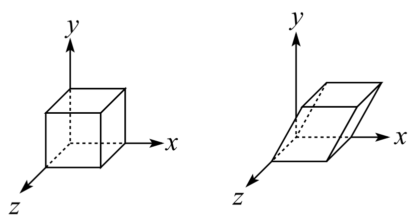


Use right hand rule

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

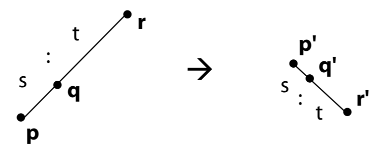


Properties of affine transformations

All of the transformations we've looked at so far are examples of "affine transformations."

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios along a line are always preserved)



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.