6. Affine transformations

Geometric transformations

Geometric transformations will map points in one space to points in another: \((x', y', z') = f(x, y, z)\).

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

We'll start in 2D...

**Representation**

We can represent a point, \(p = (x, y)\), in the plane

- as a column vector \[
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

- as a row vector \[
\begin{bmatrix}
x & y
\end{bmatrix}
\]

**Reading**

Required:
- Watt, Section 1.1.

Further reading:
- Foley, et al, Chapter 5.1-5.5.
Representation, cont.

We can represent a 2-D transformation $M$ by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $p$ is a column vector, $M$ goes on the left:

$$p' = Mp$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If $p$ is a row vector, $M^T$ goes on the right:

$$p' = pM^T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We will use column vectors.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix $M$:

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x'' = ax + by$$
$$y'' = cx + dy$$

We will develop some intimacy with the elements $a, b, c, d$...

Identity

Suppose we choose $a=d=1, b=c=0$:

- Gives the identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

Scaling

Suppose we set $b=c=0$, but let $a$ and $d$ take on any positive value:

- Gives a scaling matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- Provides differential scaling in $x$ and $y$:

$$x' = ax$$
$$y' = dy$$
Suppose we keep $b=c=0$, but let either $a$ or $d$ go negative.

Examples:

\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\quad \quad
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

Effect on unit square

Let's see how a general $2 \times 2$ transformation $M$ affects the unit square:

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\begin{bmatrix}p & q & r & s\end{bmatrix} = \begin{bmatrix}p' & q' & r' & s'\end{bmatrix}
\]

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & a & a+b & b\end{bmatrix}
\]

Effect on unit square, cont.

Observe:
- Origin invariant under $M$
- $M$ can be determined just by knowing how the corners $(1,0)$ and $(0,1)$ are mapped
- $a$ and $d$ give $x$- and $y$-scaling
- $b$ and $c$ give $x$- and $y$-shearing

Now let's leave $a=d=1$ and experiment $b$. . .

The matrix

\[
\begin{bmatrix}
1 & b \\
0 & 1
\end{bmatrix}
\]

gives:

\[
x' = x + by
\]
\[
y' = y
\]
Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for “rotation about the origin”:

Thus,

\[ M = R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

Limitations of the 2 x 2 matrix

A 2 x 2 matrix allows
- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

And then transform with a 3 x 3 matrix:

\[ \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(t) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \]

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, \( \theta \), about any point \( \mathbf{q} = [q_x, q_y]^T \) with a matrix.

Q: how would you find the matrix for rotating about \( \mathbf{q} \) by \( \theta \)?
Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, \( \theta \), about any point \( q = [qx \ qy]^T \) with a matrix:

1. Translate \( q \) to origin
2. Rotate
3. Translate back

**Note:** Transformation order is important!!

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, scaling:

\[
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} =
\begin{bmatrix}
  s_x & 0 & 0 & 0 \\
  0 & s_y & 0 & 0 \\
  0 & 0 & s_z & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]

Translation in 3D

\[
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & t_x \\
  0 & 1 & 0 & t_y \\
  0 & 0 & 1 & t_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]

Rotation in 3D

Rotation now has more possibilities in 3D:

\[
R_x(\theta) =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & \cos \theta & -\sin \theta & 0 \\
  0 & \sin \theta & \cos \theta & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
R_y(\theta) =
\begin{bmatrix}
  \cos \theta & 0 & \sin \theta & 0 \\
  0 & 1 & 0 & 0 \\
 -\sin \theta & 0 & \cos \theta & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
R_z(\theta) =
\begin{bmatrix}
  \cos \theta & -\sin \theta & 0 & 0 \\
  \sin \theta & \cos \theta & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

Use right hand rule
Shearing in 3D

Shearing is also more complicated. Here is one example:

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix}
= \begin{bmatrix}
1 & b & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

Properties of affine transformations

All of the transformations we've looked at so far are examples of “affine transformations.”

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios along a line are always preserved)

\[
\frac{p'q'}{q'r'} = \frac{pq}{qr}
\]

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.