CSEP 546: Machine Learning Autumn 2026

Jamie Morgenstern



Course Website

https://courses.cs.washington.edu/courses/csep546/26au/

Everything you need to know is there:

- Lecture slides
- Links to places to get help (Ed Discussion, staff email)
- Homeworks and due dates
- ~~Past exams to study from~~ Practice quiz generators
- Textbook and other references
- ~~Sections~~, office hours
- Grading, FAQ, etc.

Autumn 2026 logistics

Mondays, 6:30–9:20 PM Pacific

CSE2 G10

October 5, 12, 19, 26

November 2, 9, 16, 23, 30; December 7

No Monday registrar holidays in this schedule

Regular instruction ends December 11

No regular lecture December 14

Recordings: Panopto in Canvas; questions: Ed

Course Staff - Instructor

Jamie Morgenstern
Instructor, CSEP 546
Autumn 2026



Course Staff - Teaching Assistant

- *Bernie Zhu*
- *Teaching assistant, Autumn 2026*
- *Course staff: csep546-staff@cs.washington.edu*
- *Course questions: Ed Discussion in Canvas*
- *Office hours: see the course website*

CSE546: Machine Learning

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What this class is:

- **Fundamentals of ML:** bias/variance tradeoff, overfitting, optimization and computational tradeoffs, supervised learning (e.g., linear, boosting, deep learning), unsupervised models (e.g. k-means, EM, PCA)
- **Preparation for further learning:** the field is fast-moving, you will be able to apply the basics and teach yourself the latest

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What this class is not:

- **Survey course:** laundry list of algorithms, how to win Kaggle
- **An easy course:** familiarity with intro linear algebra and probability are assumed, homework will be time-consuming

Prerequisites

- Formally:
  - MATH 308, CSE 312, STAT 390 or equivalent
- Familiarity with:
  - Linear algebra
    - linear dependence, rank, linear equations, SVD
  - Multivariate calculus
  - Probability and statistics
    - Distributions, marginalization, moments, conditional expectation
  - Algorithms
    - Basic data structures, complexity
- “Can I learn these topics concurrently?”
- Use HW0 to judge skills
- **See website for review materials!**

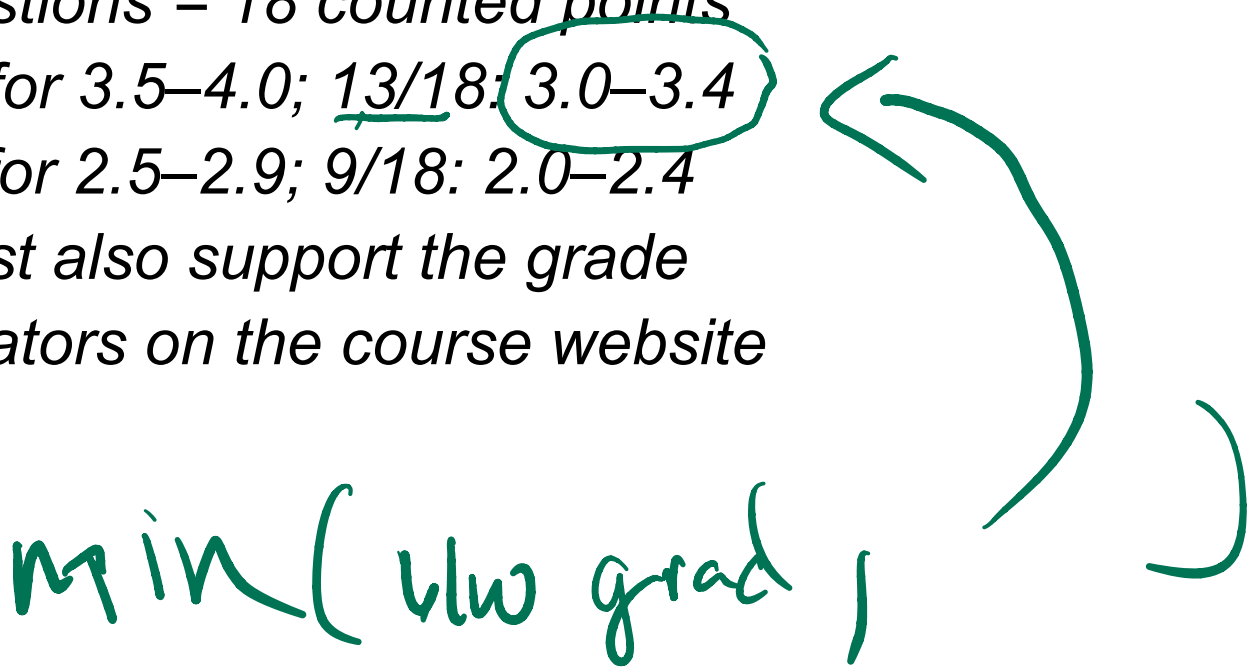
Grading

- *Homework determines your provisional grade*
- *HW0: 10%; HW1–3: 20% each; HW4: 30%*
- *Document collaboration and AI use; understand your submitted work*

- *Quiz performance sets the highest grade you can earn*
- *Eight in-person quizzes; three questions each*
- *Drop the two lowest quizzes, including absences*
- *Best six quizzes count: 18 points total*

Quiz grade eligibility

- *8 quizzes × 3 questions = 24 points offered*
- *Drop 2 whole quizzes, including zeros for absences*
- *Best 6 × 3 questions = 18 counted points*
- *15/18: eligible for 3.5–4.0; 13/18: 3.0–3.4*
- *11/18: eligible for 2.5–2.9; 9/18: 2.0–2.4*
- *Homework must also support the grade*
- *Practice generators on the course website*



Independent understanding

- *Quizzes are the expected route*
- *Complete independently: no notes, devices or AI*
- *Only a best-six score below 9/18 permits a final*
- *The optional final covers the quarter's essential concepts*
- *Its result replaces the quiz result, even if lower*
- *At 9/18 or above, no final to improve quiz eligibility*
- *Approved accommodations and required exceptions apply*

Homework

- HW 0 is out (**Due Monday, October 12, 2026 at 11:59 PM Pacific**)
  - Short *review*
  - Work individually, treat as barometer for readiness
- HW 1,2,3,4
  - They are not easy or short. Start early.
- Submit to Gradescope
- Regrade requests on Gradescope
- **There is no credit for late work**
  - **5 total late days**
  - **at most 2 days used per assignment.**

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1. All code must be written in Python
2. All written work must be typeset (e.g., LaTeX)

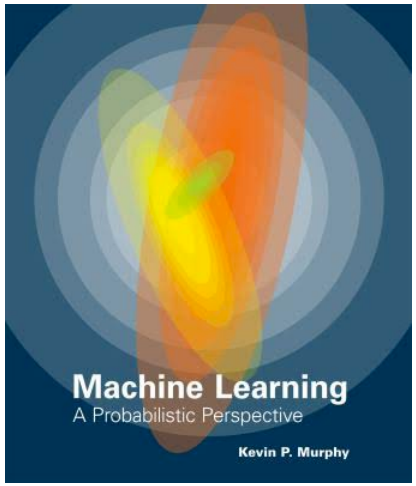
See course website for tutorials and references.

Communication Channels

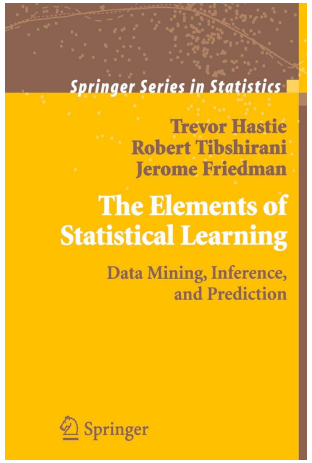
- Course questions and announcements: Ed Discussion
- Access Ed and Panopto recordings through Canvas
- Personal concerns: csep546-staff@cs.washington.edu
- Homework submission/regrades: see assignment instructions
- Office hours will be announced on Ed
- Anonymous feedback: feedback.cs.washington.edu

Textbooks

- Required Textbook:
  - ***Machine Learning: a Probabilistic Perspective*; Kevin Murphy**



- Optional Books (free PDF):
  - *The Elements of Statistical Learning: Data Mining, Inference, and Prediction;* Trevor Hastie, Robert Tibshirani, Jerome Friedman



Addcodes

- Email: Elle Brown (ellean@cs.washington.edu) for addcodes

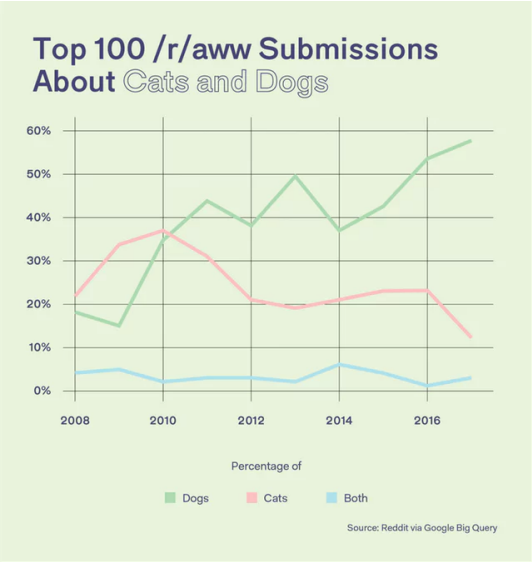
Enjoy!

- ML is becoming ubiquitous in science, engineering and beyond
- It's one of the hottest topics in industry today
- This class should give you the basic foundation for applying ML and developing new methods
- The fun begins…

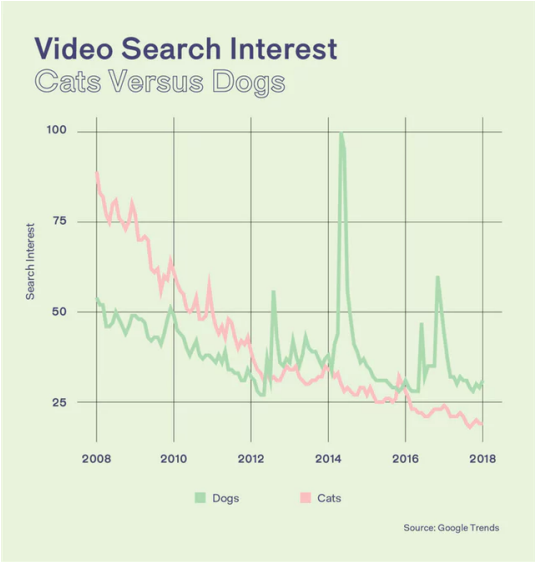
Traditional algorithms

Social media mentions of Cats vs. Dogs

Reddit



Google



Twitter?

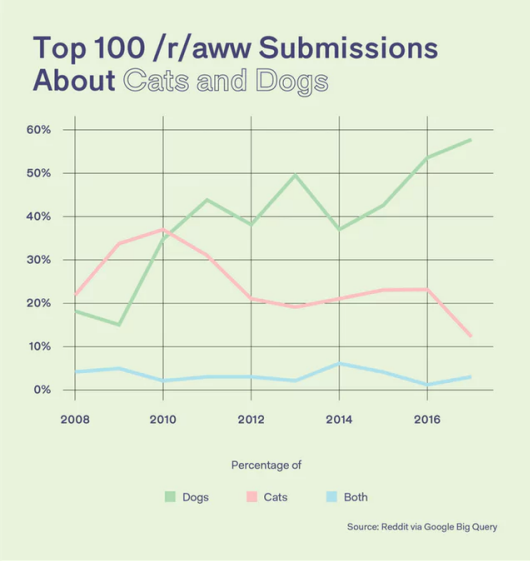
Write a program that sorts tweets into those containing "**cat**", "**dog**", or ***other***

Graphics courtesy of https://theoutline.com/post/3128/dogs-cats-internet-popularity?zd=1

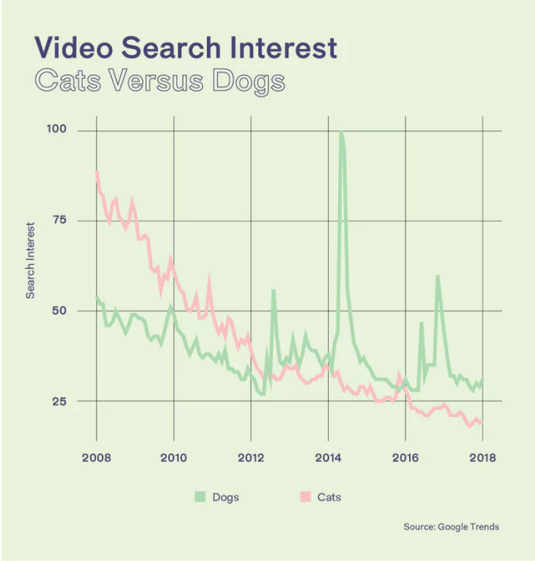
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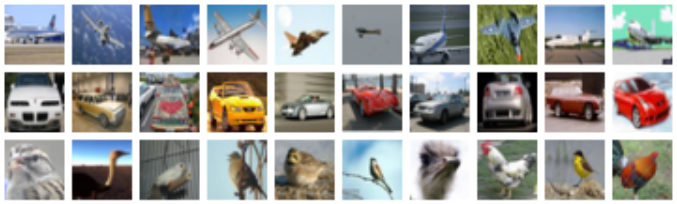
```
cats = []
dogs = []
other = []
for tweet in tweets:
    if "cat" in tweet:
        cats.append(tweet)
    elseif "dog" in tweet:
        dogs.append(tweet)
    else:
        other.append(tweet)
return cats, dogs, other
```

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Machine learning algorithms

Write a program that sorts image
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airplane

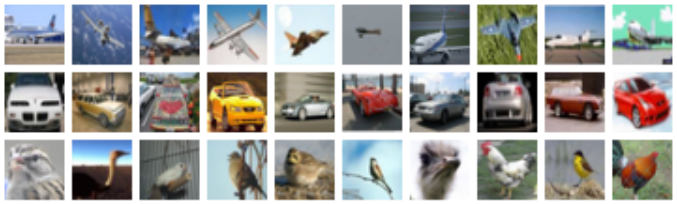
other

bird

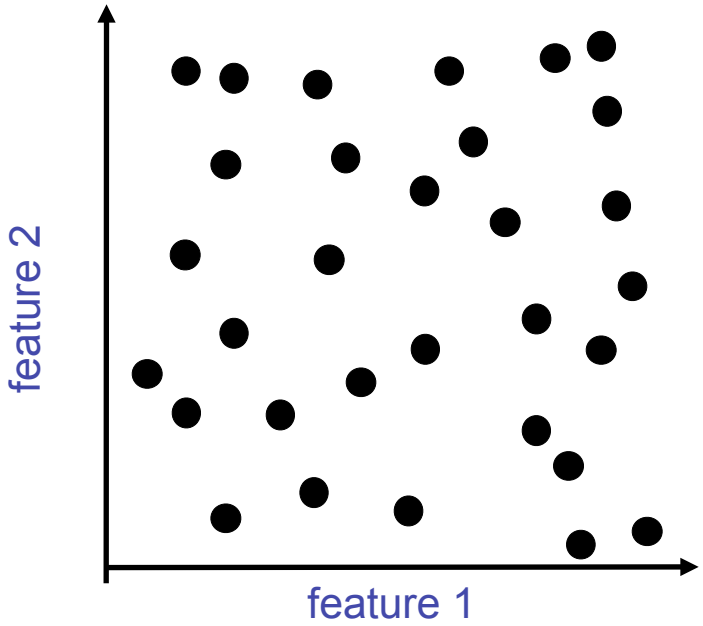
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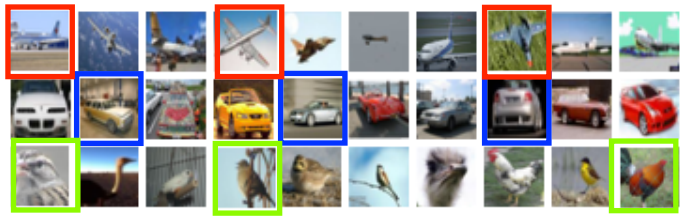
airplane
other
bird



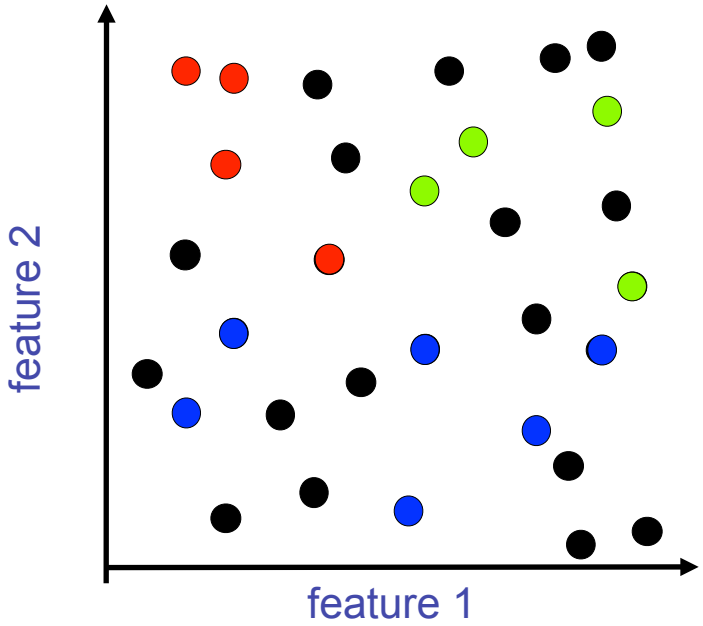

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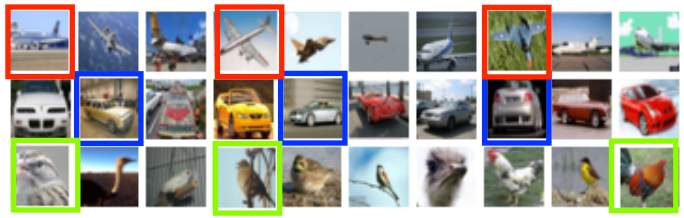
airplane
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bird



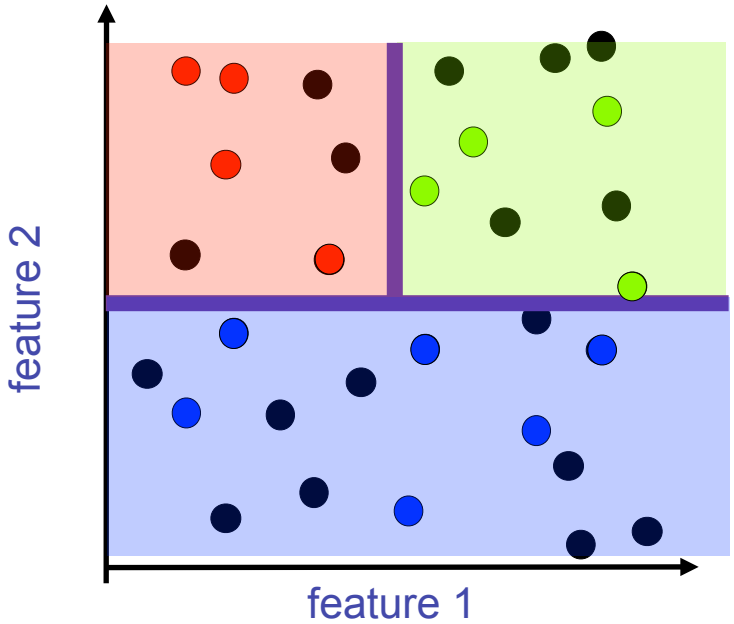

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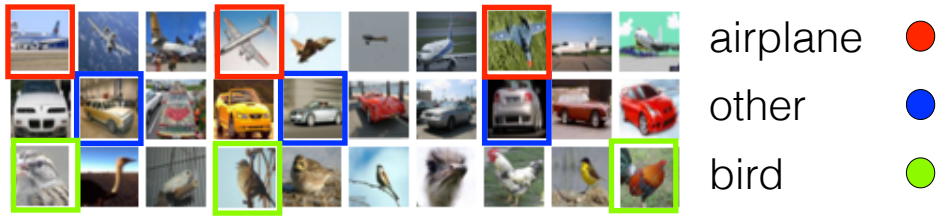
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other
bird

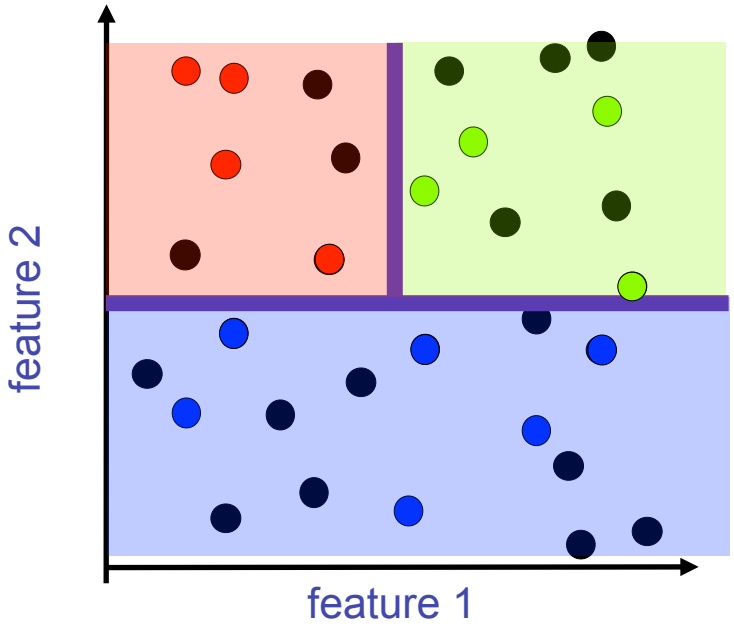



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The decision rule of
`if "cat" in tweet:`
is **hard coded by expert.**

The decision rule of
`if bird in image:`
is **LEARNED using DATA**

Machine Learning Ingredients

- **Data**: past observations
- **Hypotheses/Models**: devised to capture the patterns in data
- **Prediction**: apply model to forecast future observations

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You may also like…

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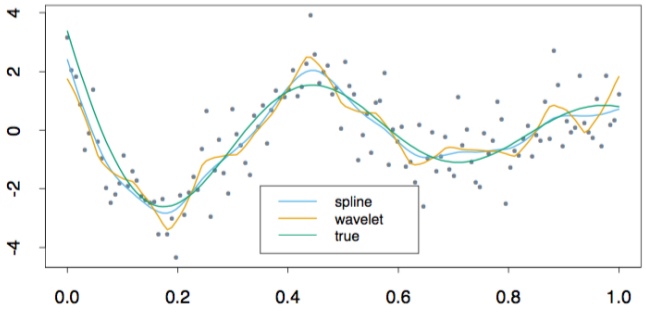


Machine learning is incredibly powerful and can have significant (unintended) negative consequences on society through targeting, excluding, and misusing.

Learning objectives of this course:

- introduction to the fundamental concepts of machine learning
- analysis and implementation of machine learning algorithms
- knowing how to use machine learning responsibly and robustly

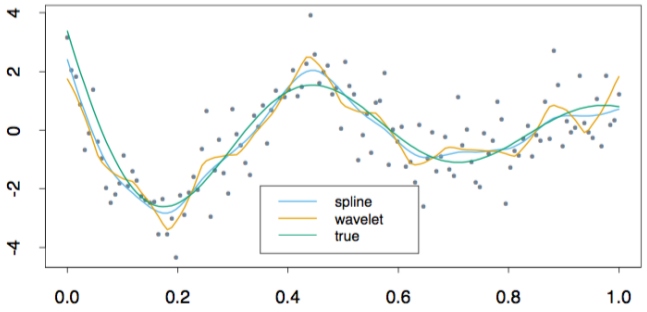
Flavors of ML



Regression

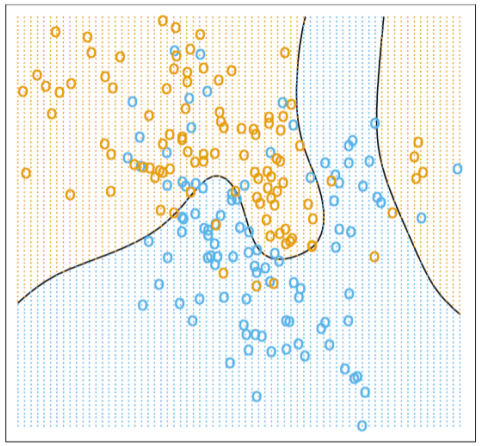
Predict continuous value:
ex: stock market, credit score, temperature, Netflix rating

Flavors of ML



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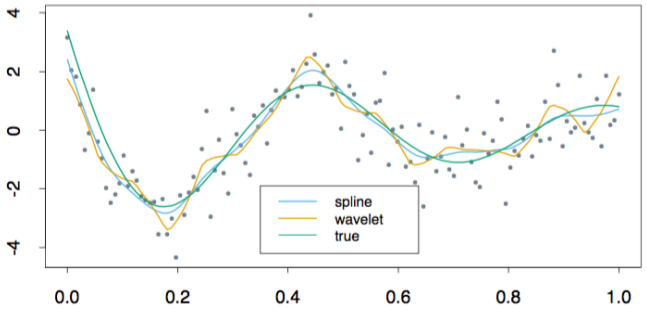
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Classification

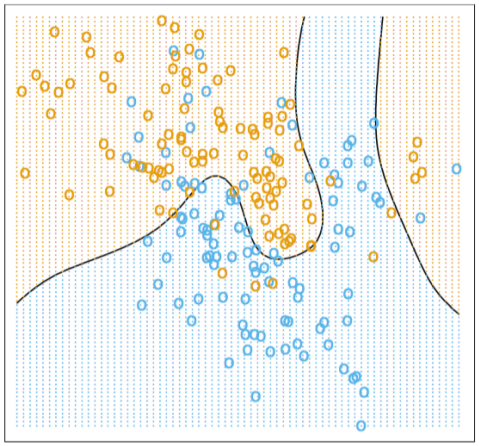
Predict categorical value:
loan or not? spam or not? what disease is this?

Flavors of ML



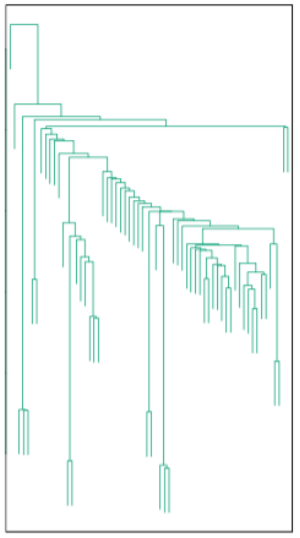
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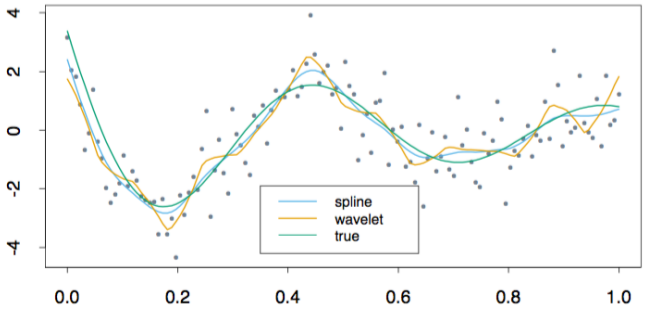
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Unsupervised Learning

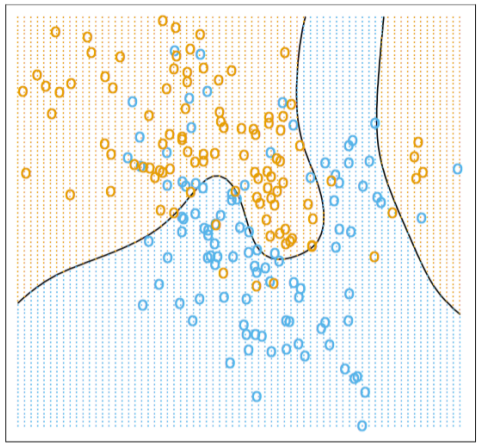
Predict structure:
tree of life from DNA, find similar images, community detection

Flavors of ML



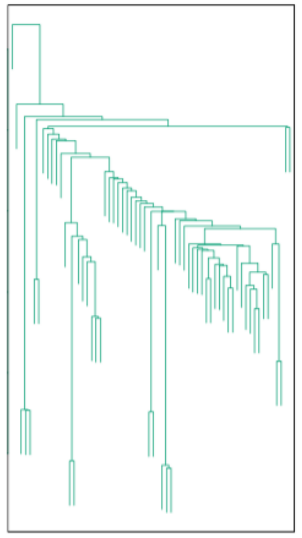
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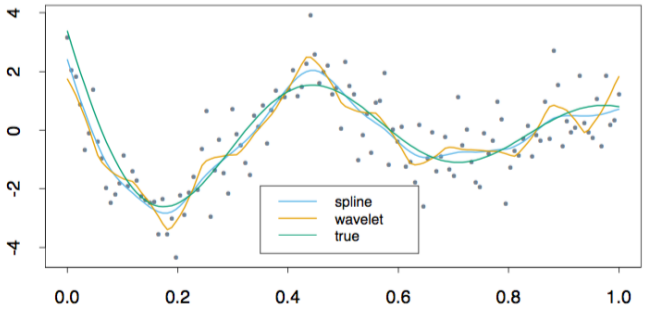


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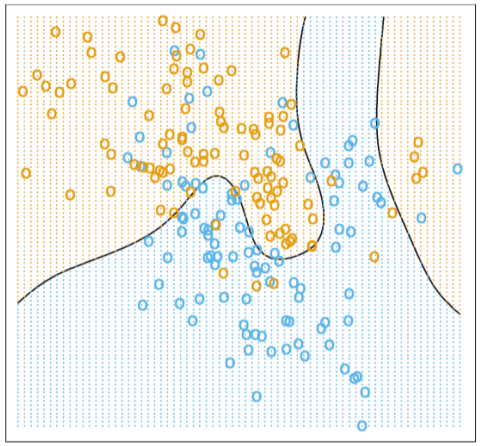
Self-supervised:
Model distribution of large-scale data, like text or images (large vision and language models)

Flavors of ML



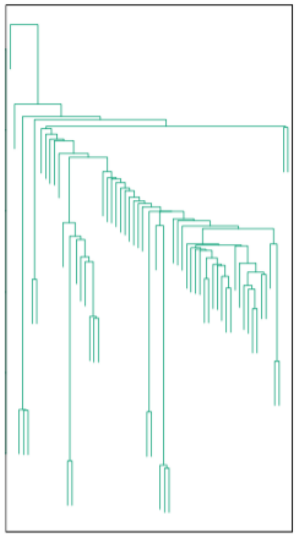
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Unsupervised Learning

Predict structure:
tree of life from DNA, find similar images, community detection

Self-supervised:
Model distribution of large-scale data, like text or images (large vision and language models)

(Generative)

Mix of statistics (conceptual) and algorithms (programming)

Probability review



Definitions

- **Random Variable**: A variable that takes on different values determined randomly.
  - Example: The height of a person from the US



- **Distribution**: The different values a random variable can take on along with the probability of that value.

- We talk about **sampling** from a distribution:
  - “Consider a sample of 100 different heights of people from the US drawn randomly from the distribution of all heights.”

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$P(X)$

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$$P(X=H) = 1/2$$
$$P(X=T) = 1/2$$

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  - "Consider a sample of 100 different heights of people from the US drawn randomly from the distribution of all heights."

Independence

Let X and Y be **random variables**

Ex. X is the outcome of the first roll of a 6-sided dice, Y is the outcome of the second roll of the dice

(X and Y take values in {1,2,3,4,5,6} each with equal probability)

An **event** is statement about the world that holds or not:

Define events $A = \{X \in \{3,4\}\}$,

$B = \{X = 1\}$,

$C = \{Y \in \{3,4\}\}$

Every event is assigned a **probability**:

$P(A) = P(X \in \{3,4\}) =$

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$P(A) = P(X \in \{3,4\}) =$ ~~2/3 = 1/6~~ 2/6 , 1/3

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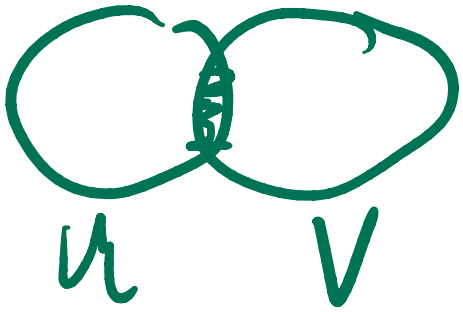
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Define events $A = \{X \in \{3,4\}\}$,

$B = \{X = 1\}$,

$C = \{Y \in \{3,4\}\}$



Every event is assigned a **probability**:

$P(A) = P(X \in \{3,4\}) = 1/3$

For any events U, V we have $P(U \cup V) = P(U) + P(V) - P(U \cap V)$

Independence

Let X and Y be **random variables**

Ex. X is the outcome of the first roll of a 6-sided dice, Y is the outcome of the second roll of the dice

(X and Y take values in {1,2,3,4,5,6} each with equal probability)

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Any events U, V are **independent** if $P(U \cap V) = P(U)P(V)$

Are A, B independent?

$P[A \cap B] = 0$ $P(A) = 1/3$ $P(B) = 1/6$

B, C?

$P(B \cap C) = 1/18$

A, C?

Independence

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Any events U, V are **independent** if $P(U \cap V) = P(U)P(V)$

Are A, B independent? $P(A)=$

B, C?

A, C?

Independence

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B, C?

A, C?

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$P(A \cap B) = 0$

B, C?

A, C?

Independence

Let X and Y be **random variables**

Ex. X is the outcome of the first roll of a 6-sided dice, Y is the outcome of the second roll of the dice

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B, C?

A, C?

Independence

Let X and Y be **random variables**

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B, C? $P(C) =$

A, C?

Independence

Let X and Y be **random variables**

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Are A, B independent? $P(A) = 1/3$ $P(B) = 1/6$ $P(A \cap B) = 0$ *No*

B, C? $P(C) = 1/3$

A, C?

Independence

Let X and Y be **random variables**

Ex. X is the outcome of the first roll of a 6-sided dice, Y is the outcome of the second roll of the dice

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B, C? $P(C) = 1/3$ $P(B) =$

A, C?

Independence

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B, C? $P(C) = 1/3$ $P(B) = 1/6$

A, C?

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$P(A \cap B) = 0$

No

B, C? $P(C) = 1/3$ $P(B) = 1/6$

$P(A \cap B) =$

A, C?

Independence

Let X and Y be **random variables**

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Are A, B independent? $P(A) = 1/3 \quad P(B) = 1/6$

$P(A \cap B) = 0$

No

B, C? $P(C) = 1/3 \quad P(B) = 1/6$

$P(A \cap B) = 1/3 \times 1/6$

A, C?

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A, C?

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No

B, C? $P(C) = 1/3 \quad P(B) = 1/6$

$P(A \cap B) = 1/3 \times 1/6$

Yes

A, C?

Yes

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We define the **conditional probability** of event U given V as

$$P(U \mid V) = \frac{P(U \cap V)}{P(V)}$$

if $U, V \perp$, $P(U \cap V) = P(U)P(V)$

What is $P(X \leq 4 \mid X \geq 3)$? $= \frac{1/3}{2/3} = \frac{1}{2}$

Independence

Let X and Y be **random variables**

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What is $P(X \leq 4 \mid X \geq 3) = \frac{P(3 \leq X \leq 4)}{P(X \geq 3)}$

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$$P(U \mid V) = \frac{P(U \cap V)}{P(V)}$$

What is $P(X \leq 4 \mid X \geq 3) = \frac{P(3 \leq X \leq 4)}{P(X \geq 3)} = \frac{1/3}{2/3} = 1/2$

Independence

Let X and Y be **random variables**

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We define the **conditional probability** of event U given V as

$$P(U \mid V) = \frac{P(U \cap V)}{P(V)}$$

Observe: if U, V are independent then $P(U \mid V) = P(U)$.
In words: if independent, V tells you nothing about U (and vice versa)

Independence

Let X and Y be **random variables**

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If independent then:

Observe: if U, V are independent then $P(U \mid V) = P(U)$.
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We define the **conditional probability** of event U given V as

$$P(U \mid V) = \frac{P(U \cap V)}{P(V)}$$

If independent then: $= \frac{P(\mathrm{U})P(\mathrm{V})}{P(\mathrm{V})}$

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If independent then: $= \frac{P(U)P(V)}{P(V)} = P(U)$

Observe: if U, V are independent then $P(U \mid V) = P(U)$.

In words: if independent, V tells you nothing about U (and vice versa)

Mean, variance

Mean $\mathbb{E}[X], \mu$

The expected value of X, each value is weighted by the probability of seeing it.

$$\mathbb{E}[X] = \sum_x P(X = x)x$$

Variance $\mathrm{Var}(X), \sigma^2$

The expected squared deviation of X from its mean.

$$\mathbb{E}[(X - \mathbb{E}[X])^2]$$

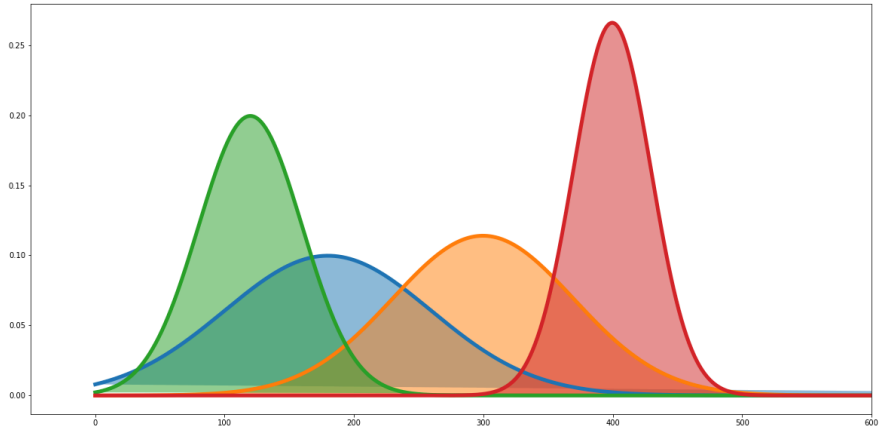
Median M

The value of X that is separating the higher half of it's range from the lower half.

$$P(X \leq M) = .5$$

Mean, variance

The mean is a prediction of the value of the random variable. Answers the question "What do I expect the height of a random person to be?"

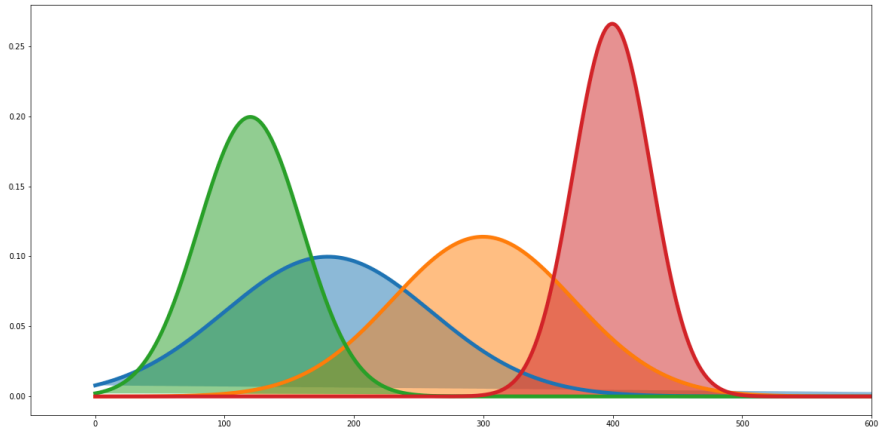


The variance captures the spread in your data. Also captures the error in the prediction using the mean. "How much do people's heights deviate?"

$$\mathbb{E}[(X - \mathbb{E}[X])^2]$$

Mean, variance

The mean is a prediction of the value of the random variable. Answers the question "What do I expect the height of a random person to be?"



Which distribution has a:

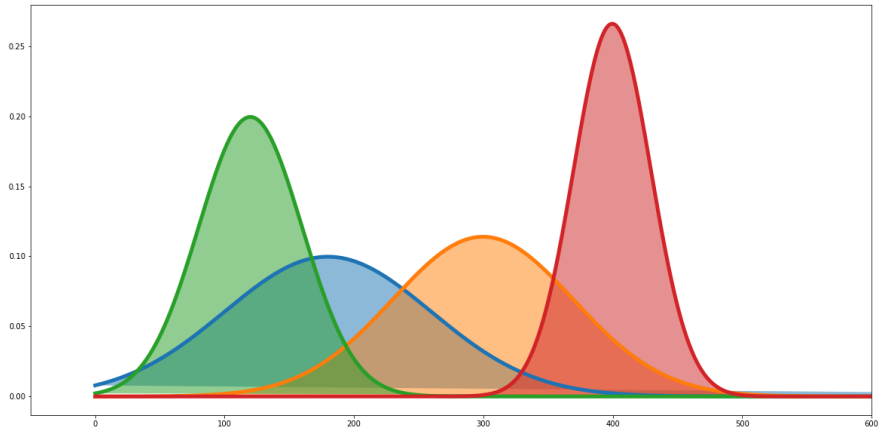
- *Large mean, small variance?*
- *Small mean, large variance?*

The variance captures the spread in your data. Also captures the error in the prediction using the mean. "How much do people's heights deviate?"

$$\mathbb{E}[(X - \mathbb{E}[X])^2]$$

Mean, variance

The mean is a prediction of the value of the random variable. Answers the question "What do I expect the height of a random person to be?"



Which distribution has a:

- *Large mean, small variance?*

  Red

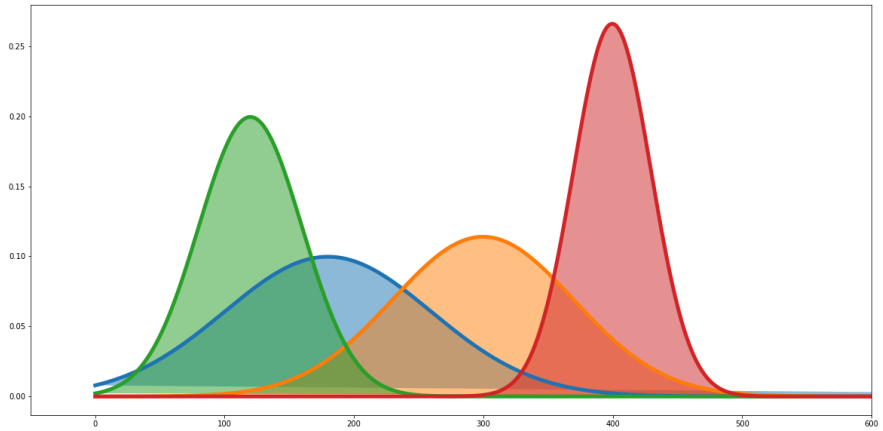
- *Small mean, large variance?*

The variance captures the spread in your data. Also captures the error in the prediction using the mean. "How much do people's heights deviate?"

$$\mathbb{E}[(X - \mathbb{E}[X])^2]$$

Mean, variance

The mean is a prediction of the value of the random variable. Answers the question "What do I expect the height of a random person to be?"



Which distribution has a:

- *Large mean, small variance?*

  Red

- *Small mean, large variance?*

  Blue

The variance captures the spread in your data. Also captures the error in the prediction using the mean. "How much do people's heights deviate?"

$$\mathbb{E}[(X - \mathbb{E}[X])^2]$$

Maximum Likelihood Estimation



Your first consulting job

- *Client*: I have special coin, if I flip it, what's the probability it will be heads?
- *You (a machine learner)*: I need to collect **data.**

- *You (a frequentist)*: The probability is:

Your first consulting job

- *Client*: I have special coin, if I flip it, what's the probability it will be heads?
- *You (a machine learner)*: I need to collect **data.**

H H

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1

Your first consulting job

- *Client*: I have special coin, if I flip it, what's the probability it will be heads?
- *You (a machine learner)*: I need to collect **data.**

H H

- *You (a frequentist)*: The probability is:

100% heads!

Your first consulting job

- *Client*: Uhhhh…. You sure about that? I just got a tails.
- *You (a machine learner)*: I need to collect **more data.**
  - *flips coin 5 times, get HHTHT

- *You*: The probability is: 60% Heads, 40% Tails!

Your first consulting job

- *Client*: Uhhhh…. You sure about that? I just got a tails.
- *You (a machine learner)*: I need to collect **more data.**
  - *flips coin 10000 times, it comes up Heads 60% of the time

- *You*: The probability is: 60% Heads, 40% Tails!

- *Client*: **Why should I believe you?**

- *You (a machine learner)*: Let's do some math!

Coin – Bernoulli Distribution

- **Data**: sequence *D= (HHTHT…),* **k heads** out of **n flips**
- **Hypothesis:** P(Heads) = θ, P(Tails) = 1-θ
  - Flips are i.i.d.:
    - Independent events
    - Identically distributed according to Bernoulli distribution

- $P(\mathcal{D}|\theta) =$

Coin – Bernoulli Distribution

- **Data**: sequence *D= (HHTHT…),* **k heads** out of **n flips**
- **Hypothesis:** P(Heads) = θ, P(Tails) = 1-θ
  - Flips are i.i.d.:
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- $P(\mathcal{D}|\theta) =$

= *P(HHTHT* | θ)

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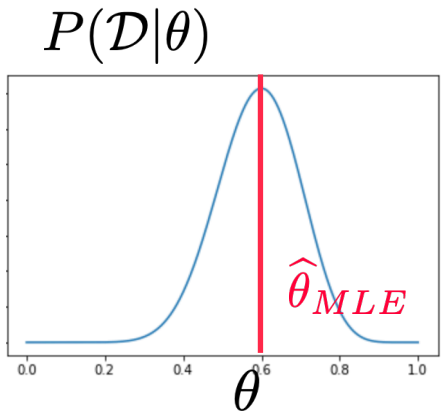
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monotonicity

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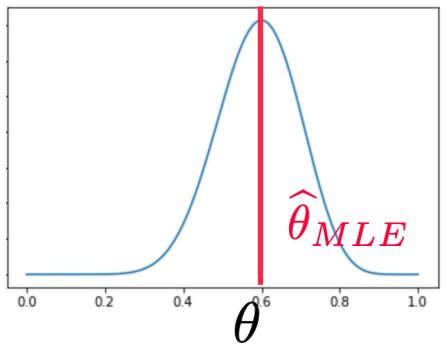
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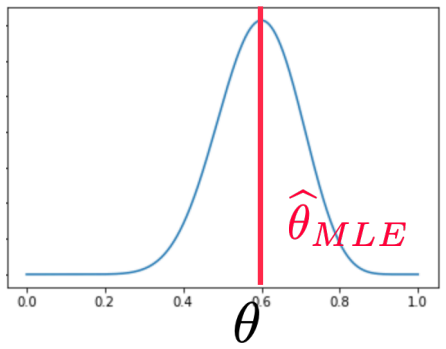
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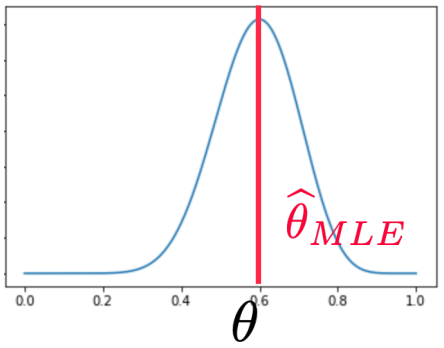
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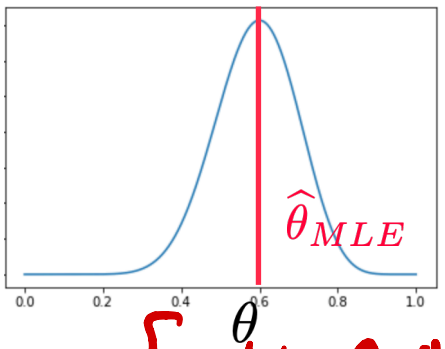
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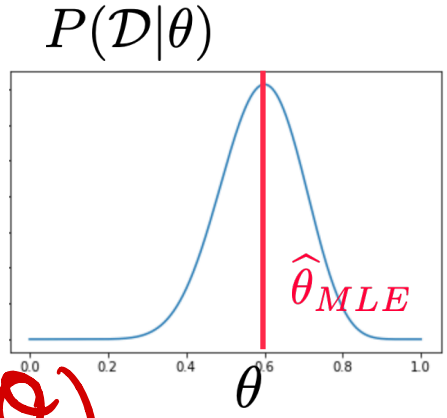


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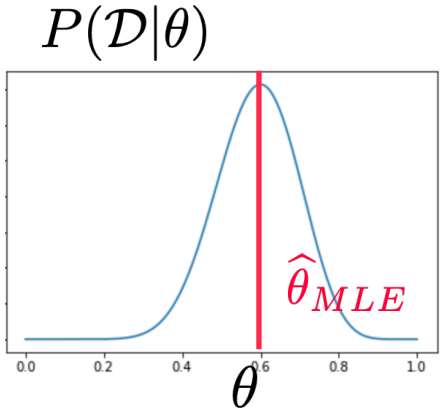
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Your first consulting job

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- We treat MLE $\widehat{\theta}_{\text{MLE}}$ as a random variable, where there is a ground truth parameter θ^* that generates the data $\mathscr{D} = (HHTTH\ldots)$ of a fixed size n

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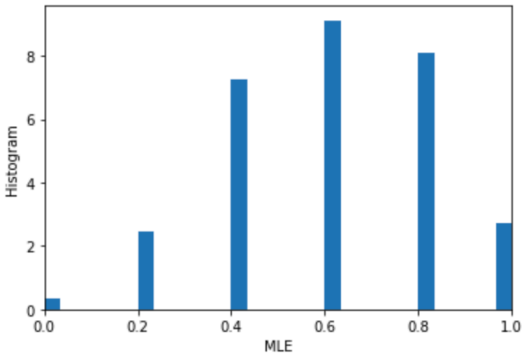
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How many flips do I need?

$$\widehat{\theta}_{MLE} = \frac{k}{n}$$

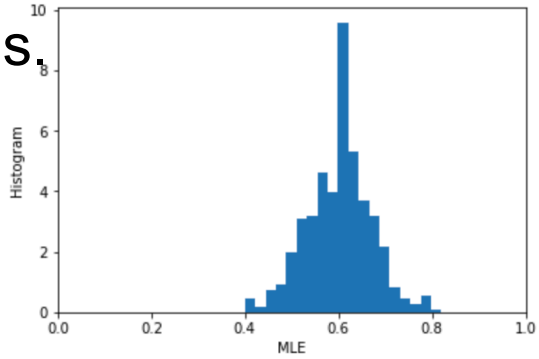
- *Client*: I flipped the coin 5 times and got 2 heads.

$$\widehat{\theta}_{MLE} = \frac{2}{5}$$



- *Client*: I flipped the coin 50 times and got 30 heads.

$$\widehat{\theta}_{MLE} = \frac{3}{5}$$



- *Client:* they are both unbiased, which one is right? Why?

Quantifying Uncertainty

- The **Variance** is the expected squared deviation from the mean:

$$\text{Variance}(\widehat{\theta}_{MLE}) := \mathbb{E}\left[\left(\widehat{\theta}_{MLE} - \mathbb{E}[\widehat{\theta}_{MLE}]\right)^2\right]$$

- As a rule of thumb

$$\widehat{\theta}_{\text{MLE}} \simeq \mathbb{E}\left[\widehat{\theta}_{\text{MLE}}\right] \pm \sqrt{\text{Variance}(\widehat{\theta}_{\text{MLE}})}$$

- Second good property of MLE: **minimum (asymptotic) variance**
- **Exercise**: compute the $\text{Variance}(\widehat{\theta}_{\text{MLE}})$

Expectation versus High Probability

- Tail bound of a random variable
- For any ε>0 can we bound $\mathbb{P}(|\widehat{\theta}_{MLE} - \mathbb{E}[\widehat{\theta}_{MLE}]| \geq \epsilon)$?

Markov's inequality

For any $t > 0$ and non-negative random variable X

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$$

- **Exercise**: Apply Markov's inequality to obtain bound. (Hint: set $X = |\widehat{\theta}_{MLE} - \theta^*|^2$)

Maximum Likelihood Estimation

Observe $X_1, X_2, \ldots, X_n$ drawn IID from $f(x;\theta)$ for some "true" $\theta = \theta_*$

Likelihood function $L_n(\theta) = \prod_{i=1}^{n} f(X_i;\theta)$

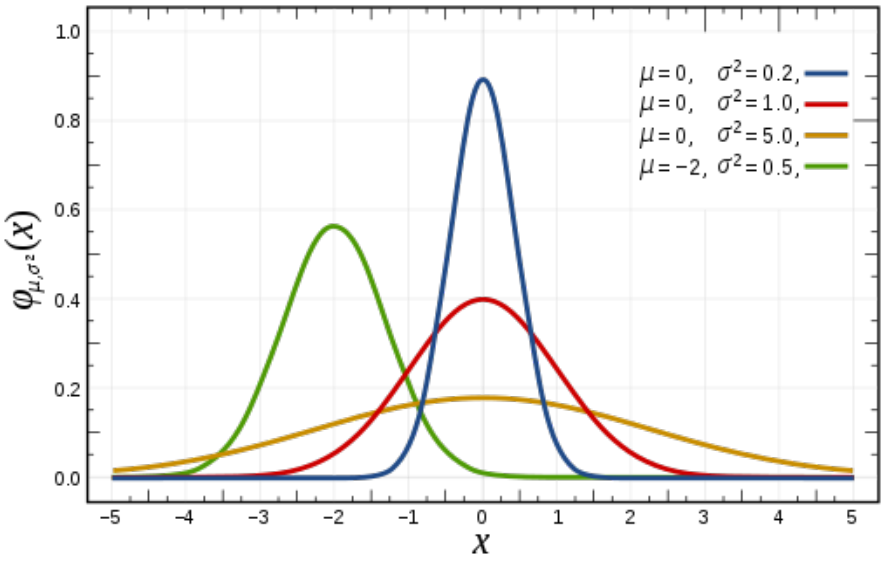
Log-Likelihood function $l_n(\theta) = \log(L_n(\theta)) = \sum_{i=1}^{n} \log(f(X_i;\theta))$

Maximum Likelihood Estimator (MLE) $\widehat{\theta}_{MLE} = \arg\max_{\theta} L_n(\theta)$

What about continuous variables?

- *Client*: What if I am measuring a **continuous variable**?
- ***You*: Let me tell you about Gaussians…**

$$P(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

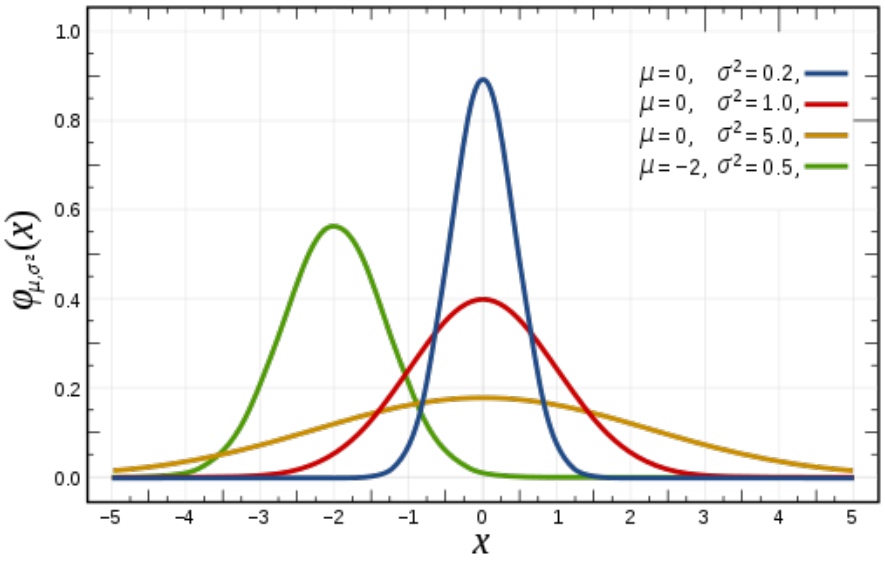


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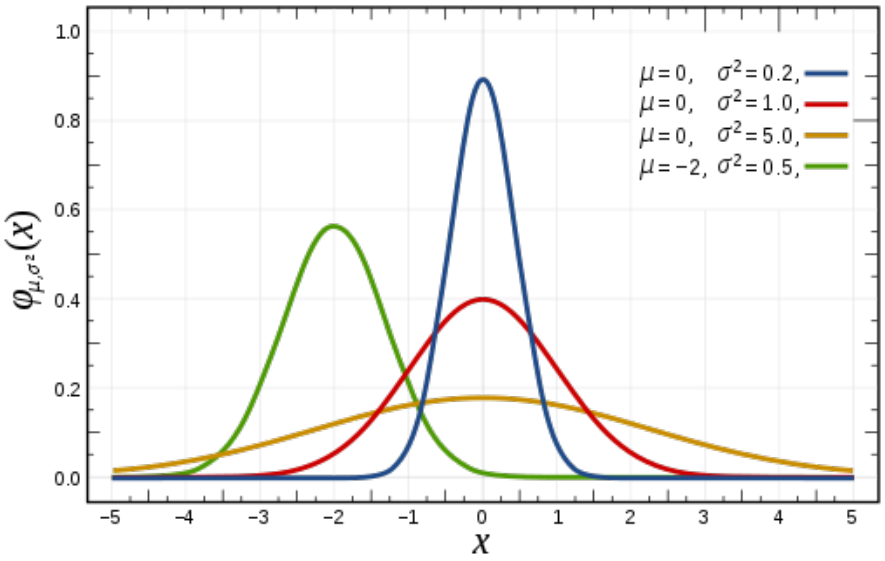
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(good candidate for formula sheet)



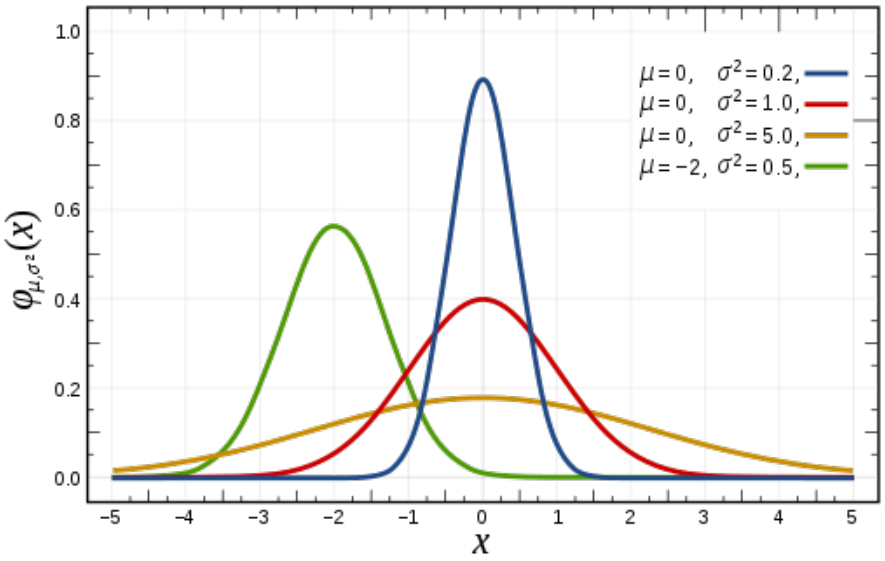
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Given a set of i.i.d. samples from a Gaussian, fit what parameters?

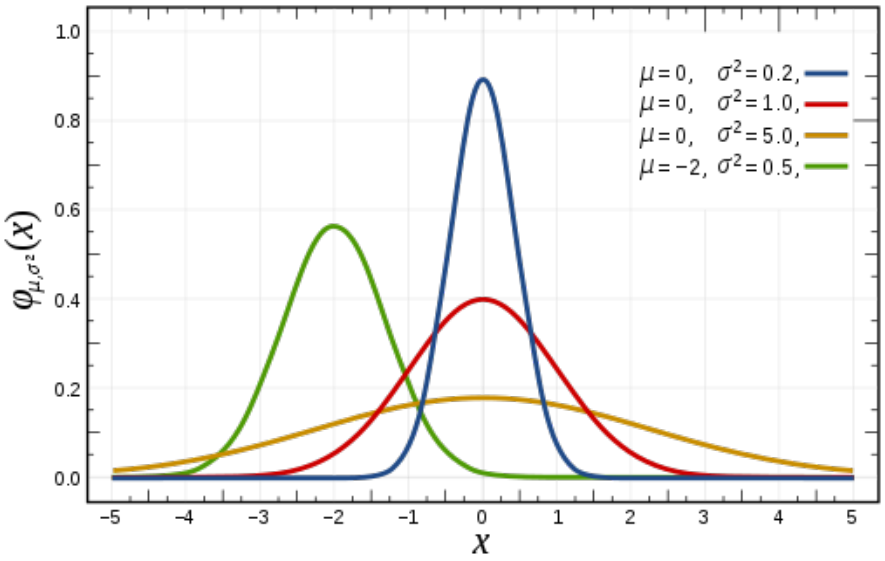
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PDF of Gaussian

(good candidate for formula sheet)



Given a set of i.i.d. samples from a Gaussian, fit what parameters?

$$\theta = [\mu, \sigma]$$

Some properties of Gaussians

- affine transformation (multiplying by scalar and adding a constant)
  - $X \sim N(\mu, \sigma^2)$
  - $Y = aX + b \quad \rightarrow \quad Y \sim N(a\mu+b, a^2\sigma^2)$

- Sum of Gaussians
  - $X \sim N(\mu_X, \sigma^2_X)$
  - $Y \sim N(\mu_Y, \sigma^2_Y)$
  - $Z = X+Y \quad \rightarrow \quad Z \sim N(\mu_X+\mu_Y, \sigma^2_X+\sigma^2_Y)$

MLE for Gaussian

- Prob. of i.i.d. samples $D=\{x_1,\ldots,x_n\}$ (e.g., temperature):

$$P(\mathcal{D}|\mu,\sigma) = P(x_1,\ldots,x_n|\mu,\sigma)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

- Log-likelihood of data:

$$\log P(\mathcal{D}|\mu,\sigma) = -n\log(\sigma\sqrt{2\pi}) - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}$$

- What is $\widehat{\theta}_{MLE}$ for $\theta = (\mu,\sigma^2)$? Draw a picture!

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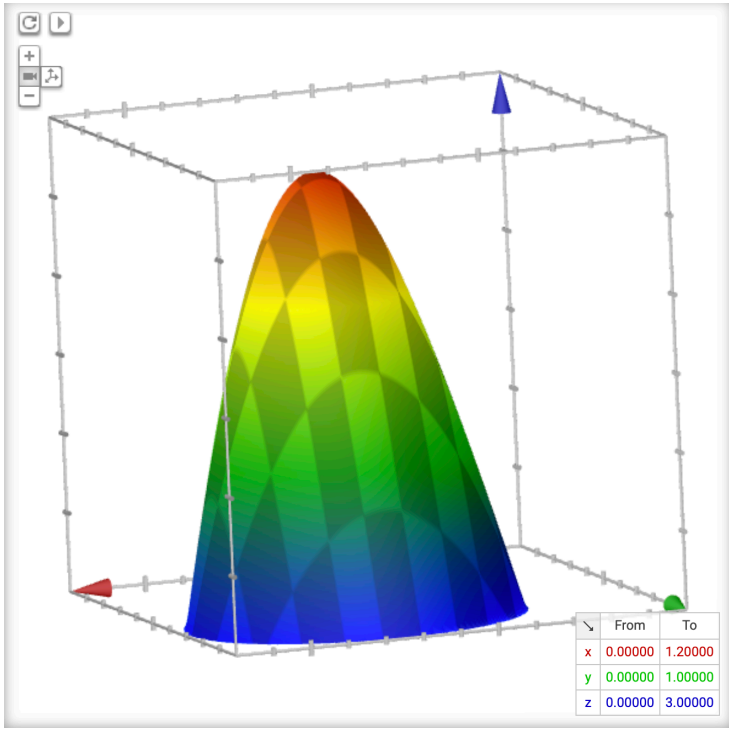
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MLE for variance

- Again, set derivative to zero:

$$\frac{d}{d\sigma}\log P(\mathscr{D};\mu,\sigma) = \frac{d}{d\sigma}\left[-n\log(\sigma\sqrt{2\pi}) - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$



MLE for Gaussian

Generate $\mathcal{D} = \{x_1, ..., x_n\}$, where

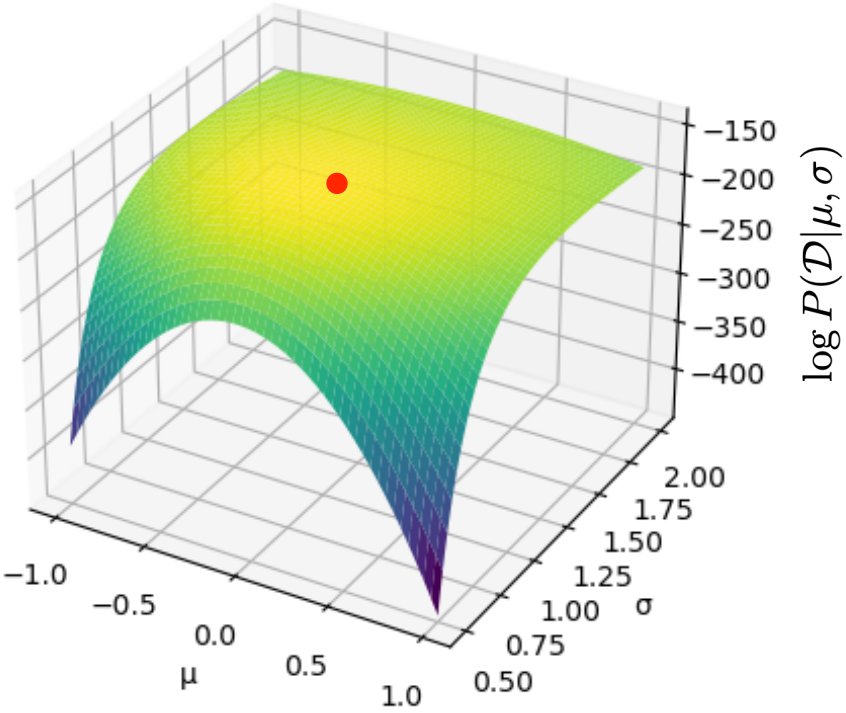
$$n = 100$$

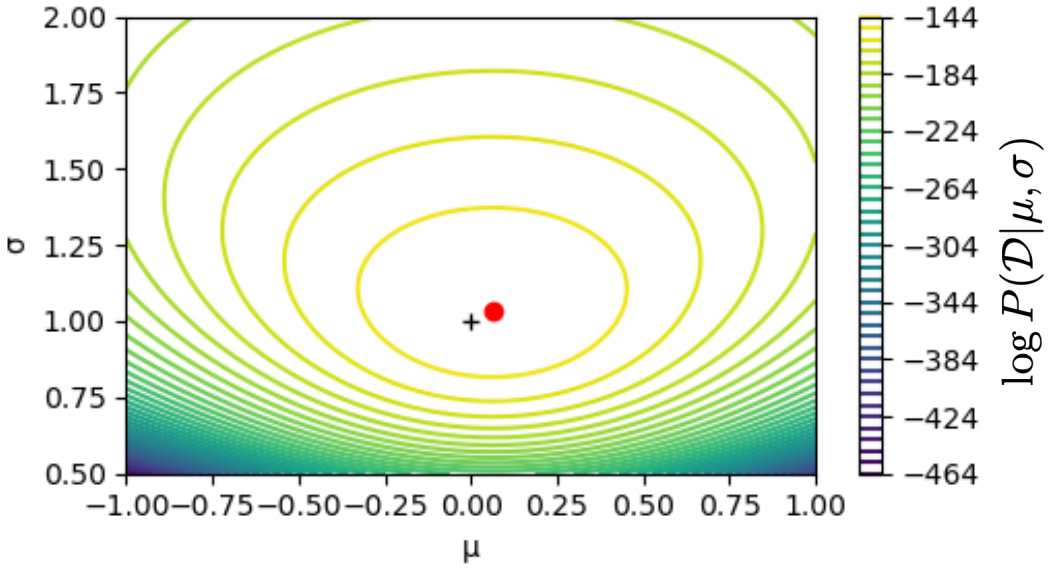
$$x_i \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = 0$$

$$\sigma^2 = 1$$

$$\log P(\mathcal{D}|\mu, \sigma) = -n \log(\sigma\sqrt{2\pi}) - \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}$$





+ $(\mu_{True}, \sigma_{True})$

● $(\hat{\mu}_{MLE}, \hat{\sigma}_{MLE})$

Your second learning algorithm: MLE for mean of a Gaussian

- What's MLE for mean? Set **partial derivative** to zero.

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empirical mean!

MLE for variance

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reminders for formulas:

$$\frac{d}{dx}\log x = \frac{1}{x}$$

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Learning Gaussian parameters

- MLE:

$$\widehat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\widehat{\sigma^2}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \widehat{\mu}_{MLE})^2$$

- MLE for the variance of a Gaussian is **biased**

$$\mathbb{E}[\widehat{\sigma^2}_{MLE}] \neq \sigma^2$$

- Unbiased variance estimator:

$$\widehat{\sigma^2}_{unbiased} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \widehat{\mu}_{MLE})^2$$

Maximum Likelihood Estimation

Observe $X_1, X_2, \ldots, X_n$ drawn IID from $f(x;\theta)$ for some "true" $\theta = \theta_*$

Likelihood function $L_n(\theta) = \prod_{i=1}^{n} f(X_i;\theta)$

Log-Likelihood function $l_n(\theta) = \log(L_n(\theta)) = \sum_{i=1}^{n} \log(f(X_i;\theta))$

Maximum Likelihood Estimator (MLE) $\widehat{\theta}_{MLE} = \arg\max_{\theta} L_n(\theta)$

Properties (under benign regularity conditions—smoothness, identifiability, etc.):

- Asymptotically consistent and normal: $\frac{\widehat{\theta}_{MLE} - \theta_*}{\widehat{se}} \sim \mathcal{N}(0,1)$
- Asymptotic Optimality, minimum variance (see Cramer-Rao lower bound)

The MLE is a "recipe" that begins with a *model* for data $f(x;\theta)$

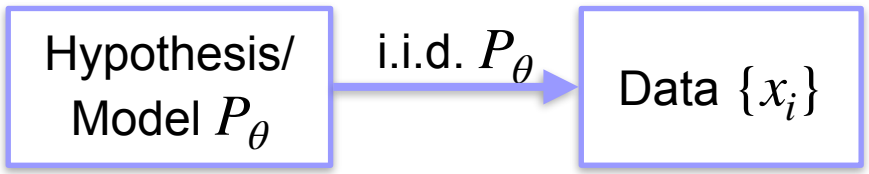
Recap

- Learning is…
  - Collect some data
    - E.g., coin flips

Data $\{x_i\}$

Recap

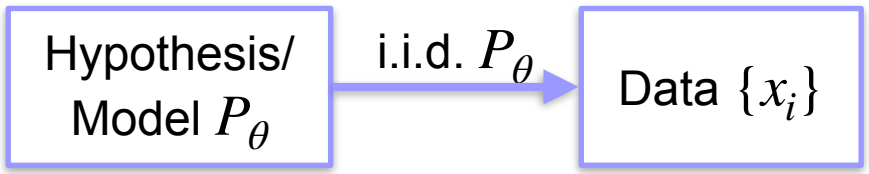
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Recap

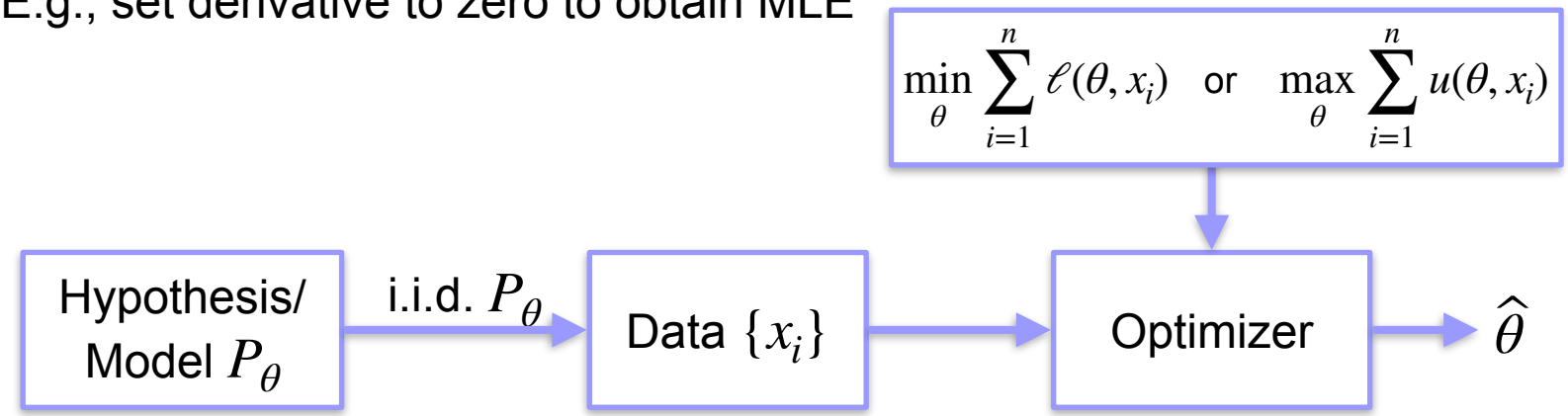
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  - Collect some data
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    - E.g., binomial
  - Choose a loss function
    - E.g., data likelihood

$$\min_{\theta} \sum_{i=1}^{n} \ell(\theta, x_i) \quad \text{or} \quad \max_{\theta} \sum_{i=1}^{n} u(\theta, x_i)$$



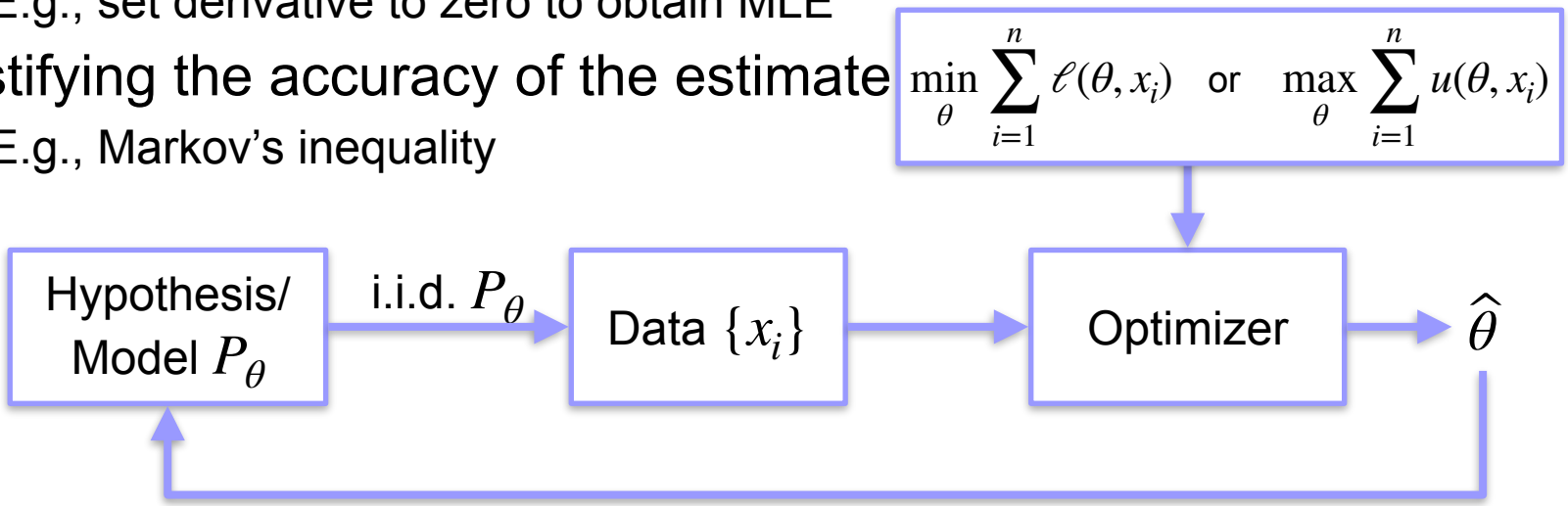
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- Learning is…
  - Collect some data
    - E.g., coin flips
  - Choose a hypothesis class or model
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  - Choose a loss function
    - E.g., data likelihood
  - Choose an optimization procedure
    - E.g., set derivative to zero to obtain MLE



Recap

- Learning is…
  - Collect some data
    - E.g., coin flips
  - Choose a hypothesis class or model
    - E.g., binomial
  - Choose a loss function
    - E.g., data likelihood
  - Choose an optimization procedure
    - E.g., set derivative to zero to obtain MLE
  - Justifying the accuracy of the estimate
    - E.g., Markov's inequality



Applications preview

Maximum Likelihood Estimation

Maximum Likelihood Estimation

Why is it useful to recover the "true" parameters θ_* of a probabilistic model?

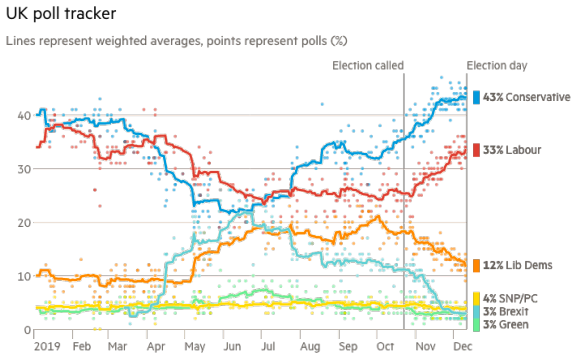
- **Estimation** of the parameters θ_* is the goal
- Help **interpret** or summarize large datasets
- Make **predictions** about future data
- **Generate** new data $X \sim f(\cdot\,; \hat{\theta}_{\text{MLE}})$

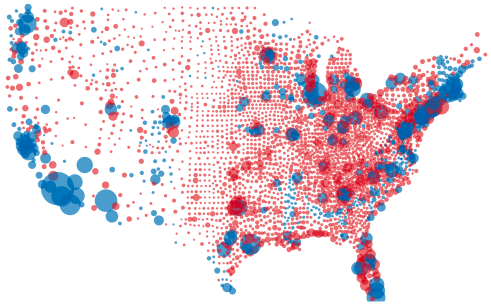
Estimation

Observe $X_1, X_2, \ldots, X_n$ drawn IID from $f(x;\theta)$ for some "true" $\theta = \theta_*$

Opinion polls
How does the greater population feel about an issue?
Correct for over-sampling?

- θ_* is "true" average opinion
- $X_1, X_2, \ldots$ are sample calls



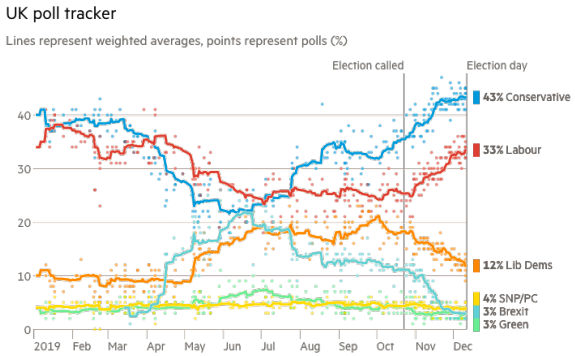


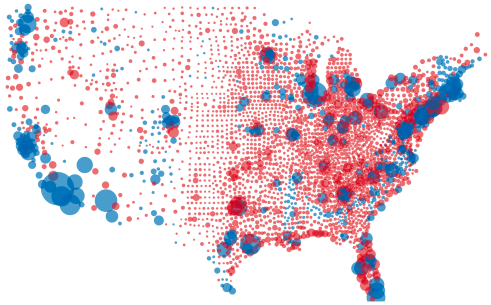
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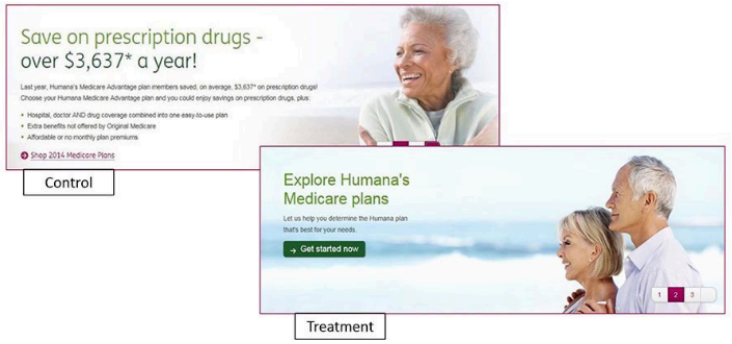
- θ_* is "true" average opinion
- $X_1, X_2, \ldots$ are sample calls





A/B testing
How do we figure out which ad results in more click-through?

- θ_* are the "true" average rates
- $X_1, X_2, \ldots$ are binary "clicks"



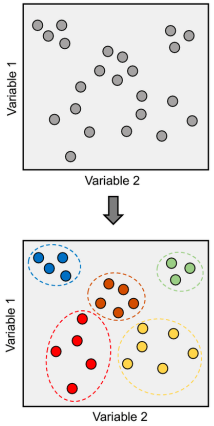
Interpret

Observe $X_1, X_2, \ldots, X_n$ drawn IID from $f(x;\theta)$ for some "true" $\theta = \theta_*$

Customer segmentation / clustering
Can we identify distinct groups of customers by their behavior?

- θ_* describes "center" of distinct groups
- $X_1, X_2, \ldots$ are individual customers





Interpret

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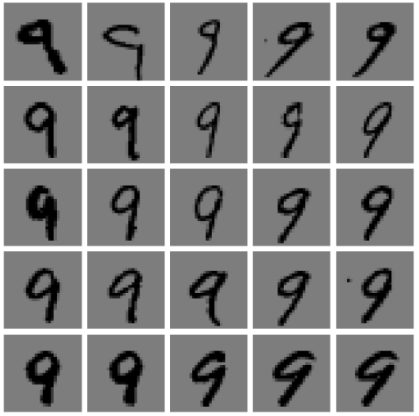
Customer segmentation / clustering
Can we identify distinct groups of customers by their behavior?

- θ_* describes "center" of distinct groups
- $X_1, X_2, \ldots$ are individual customers



Data exploration
What are the degrees of freedom of the dataset?

- θ_* describes the principle directions of variation
- $X_1, X_2, \ldots$ are the individual images



Predict

Observe $X_1, X_2, \ldots, X_n$ drawn IID from $f(x;\theta)$ for some "true" $\theta = \theta_*$

Content recommendation
Can we predict how much someone will like a movie based on past ratings?

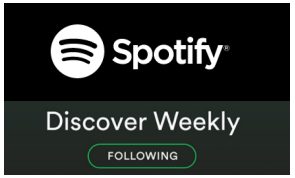
- θ_* describes user's preferences
- $X_1, X_2, \ldots$ are (movie, rating) pairs

Predict

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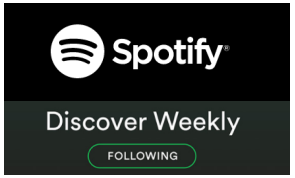
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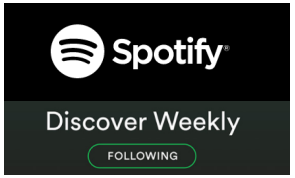
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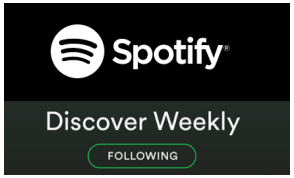
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Content recommendation
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- $X_1, X_2, \ldots$ are (movie, rating) pairs







Object recognition / classification
Identify a flower given just its picture?

- θ_* describes the characteristics of each kind of flower
- $X_1, X_2, \ldots$ are the (image, label) pairs



(a) (b) (c)

Figure 1.1: Three types of Iris flowers: Setosa, Versicolor and Virginica. Used with kind permission of Dennis Kramb and SIGNA.

index	sl	sw	pl	pw	label
0	5.1	3.5	1.4	0.2	Setosa
1	4.9	3.0	1.4	0.2	Setosa
	...				
50	7.0	3.2	4.7	1.4	Versicolor
	...				
149	5.9	3.0	5.1	1.8	Virginica

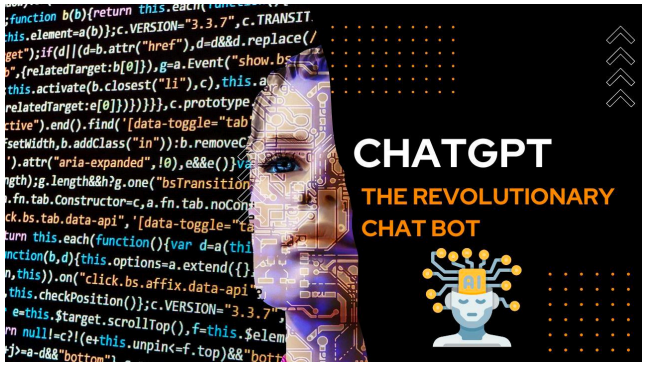
Generate

Observe $X_1, X_2, \ldots, X_n$ drawn IID from $f(x;\theta)$ for some "true" $\theta = \theta_*$

Text generation
Can AI generate text that could have been written like a human?

- θ_* describes language structure
- $X_1, X_2, \ldots$ are text snippets found online

"Kaia the dog wasn't a natural pick to go to mars. No one could have predicted she would..."



https://chat.openai.com/chat

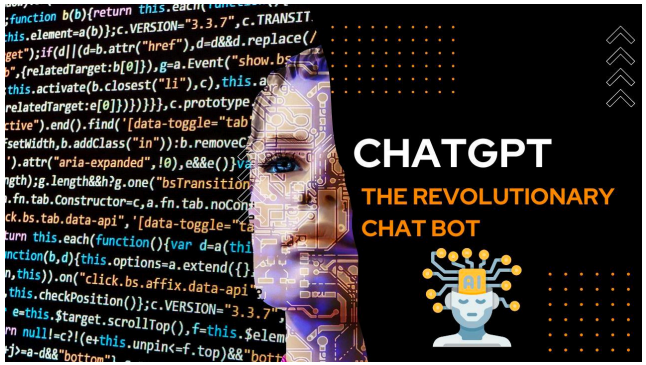
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Image to text generation
Can AI generate an image from a prompt?

- θ_* describes the coupled structure of images and text
- $X_1, X_2, \ldots$ are the (image, caption) pairs found online

"dog talking on cell phone under water, oil painting"



https://labs.openai.com/

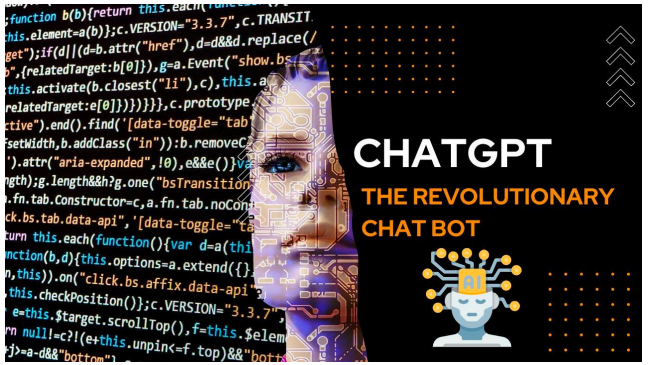
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MLE!

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Linear Regression

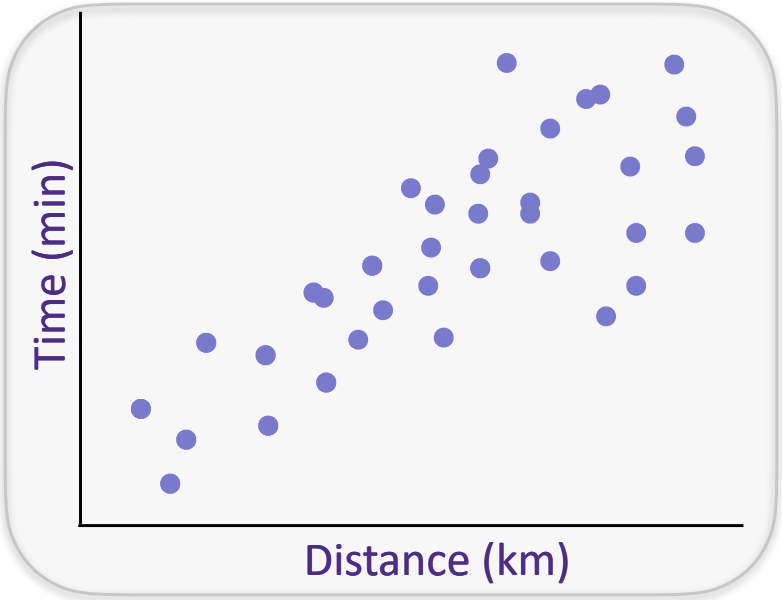


The regression problem, 1-dimensional

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)



Training Data:

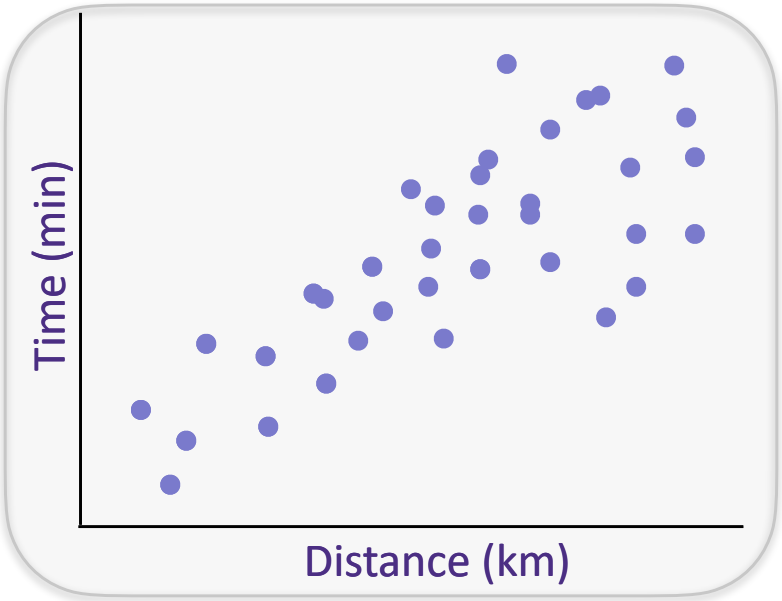
$$\{(x_i, y_i)\}_{i=1}^{n} \qquad x_i \in \mathbb{R} \quad y_i \in \mathbb{R}$$

The regression problem, 1-dimensional

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)



previously we just had x

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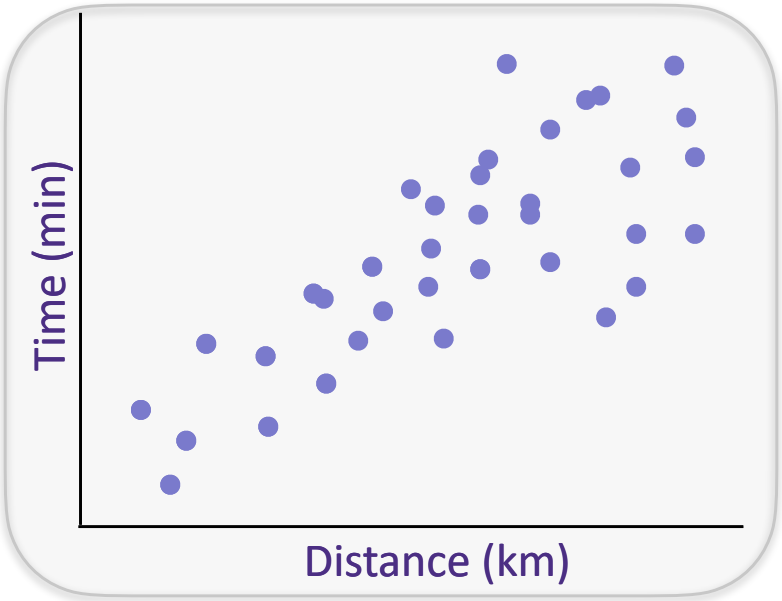
Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)

now we have label y

previously we just had x



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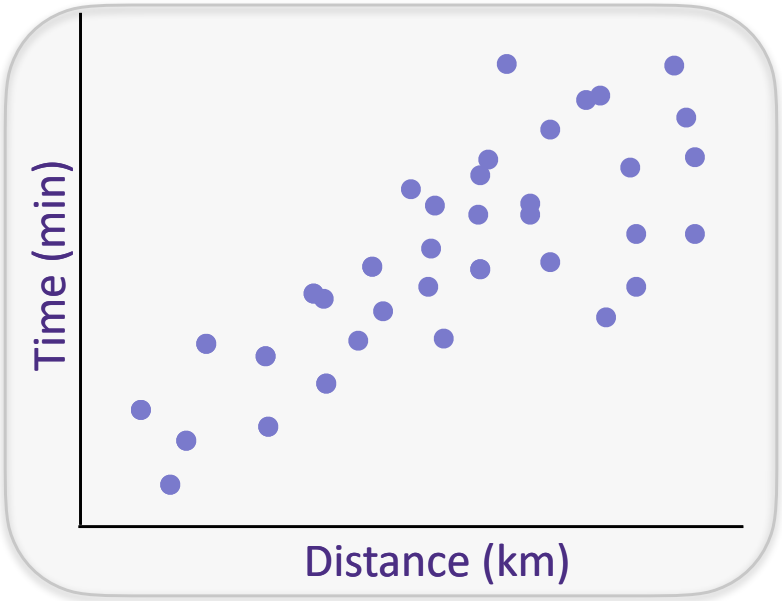
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x = distance (km)

Goal: learn p(y|x)

now we have label y

previously we just had x



Training Data:

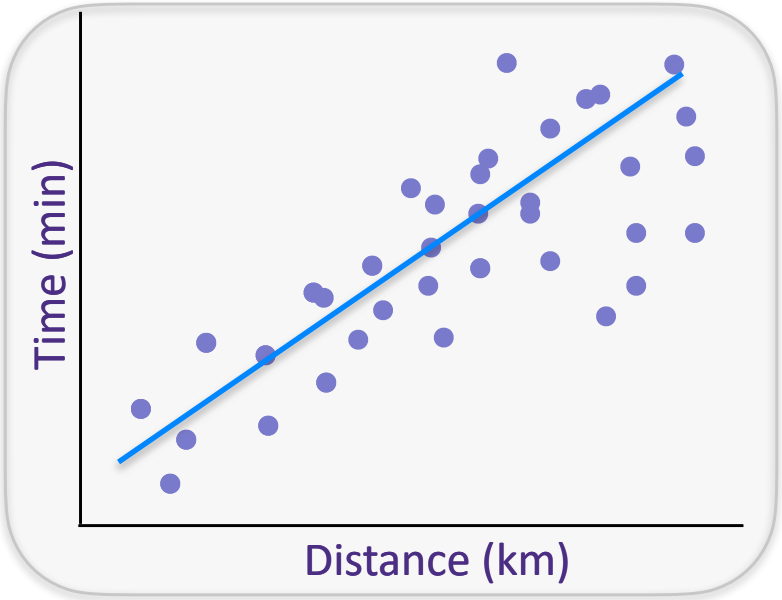
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The regression problem, 1-dimensional

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)



Goal: learn p(y|x)

now we have label y

previously we just had x

Training Data:

$$\{(x_i, y_i)\}_{i=1}^{n} \qquad \begin{matrix} x_i \in \mathbb{R} \\ y_i \in \mathbb{R} \end{matrix}$$

The regression problem, 1-dimensional

Potato delivery logs, predict:

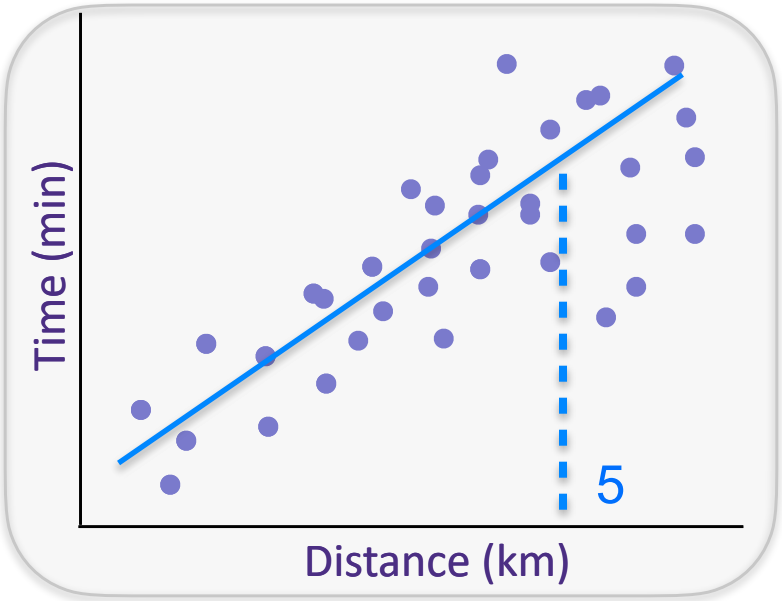
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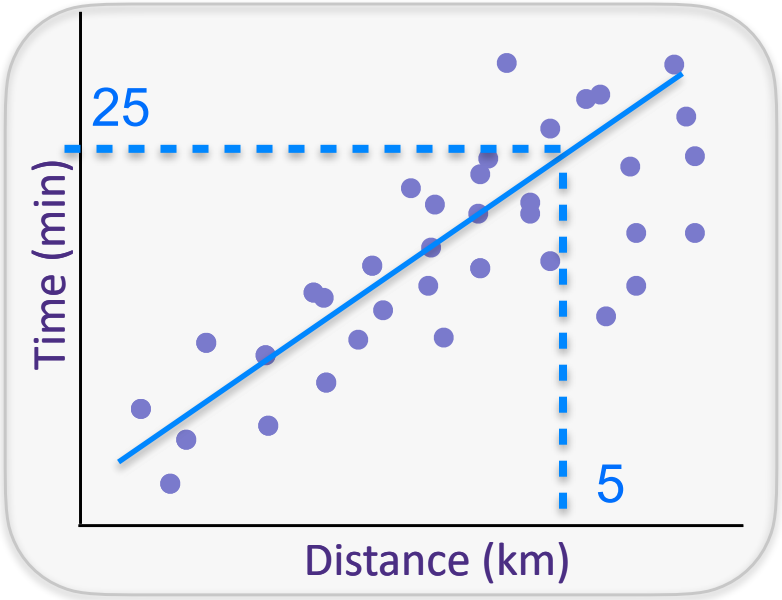
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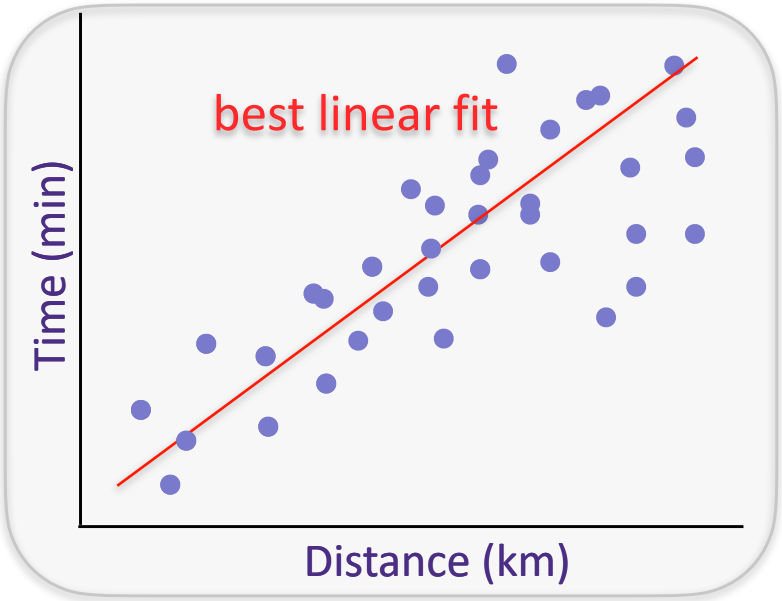
$$\{(x_i, y_i)\}_{i=1}^{n} \qquad x_i \in \mathbb{R} \quad y_i \in \mathbb{R}$$

Fit a function to our data, 1-d

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)



Training Data: $\{(x_i, y_i)\}_{i=1}^{n}$ $x_i \in \mathbb{R}$ $y_i \in \mathbb{R}$

Hypothesis/Model: linear

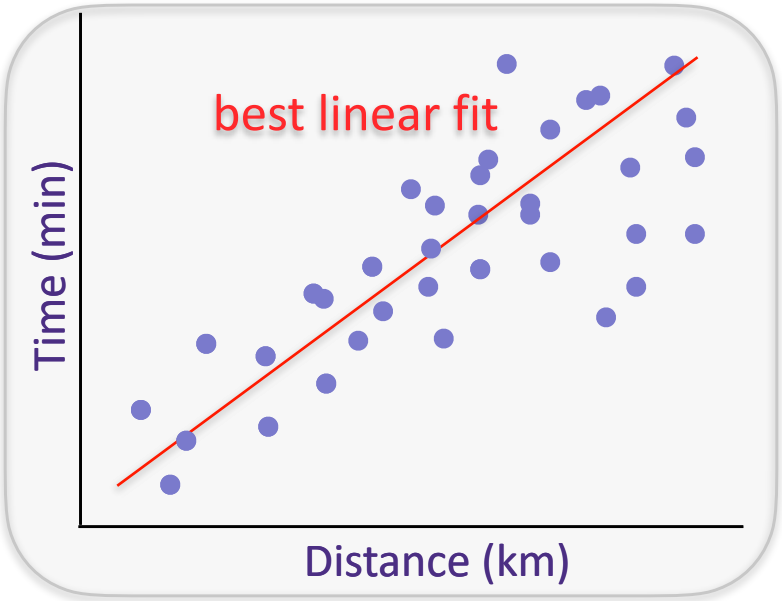
$$y_i = x_i w + \epsilon_i \qquad \epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$$

Fit a function to our data, 1-d

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)



Training Data: $\{(x_i, y_i)\}_{i=1}^{n}$ $x_i \in \mathbb{R}$ $y_i \in \mathbb{R}$

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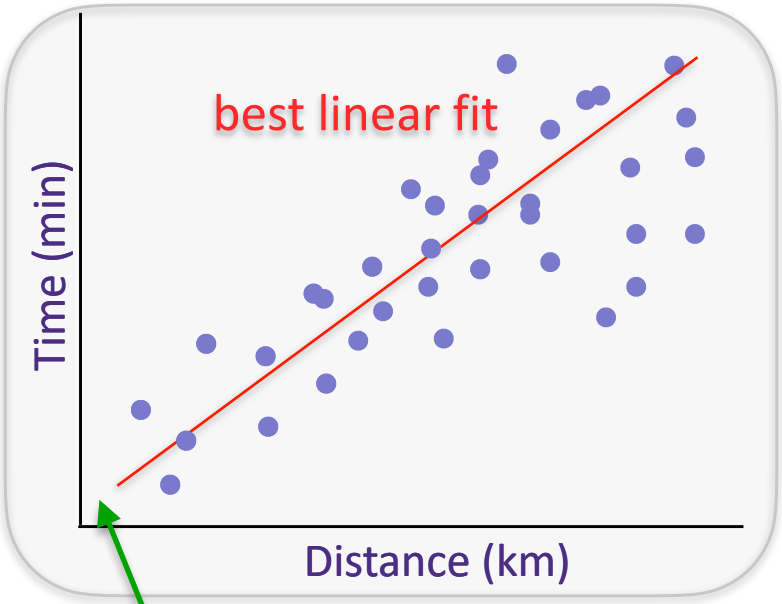
$w \in \mathbb{R}$ *slope of line*

Fit a function to our data, 1-d

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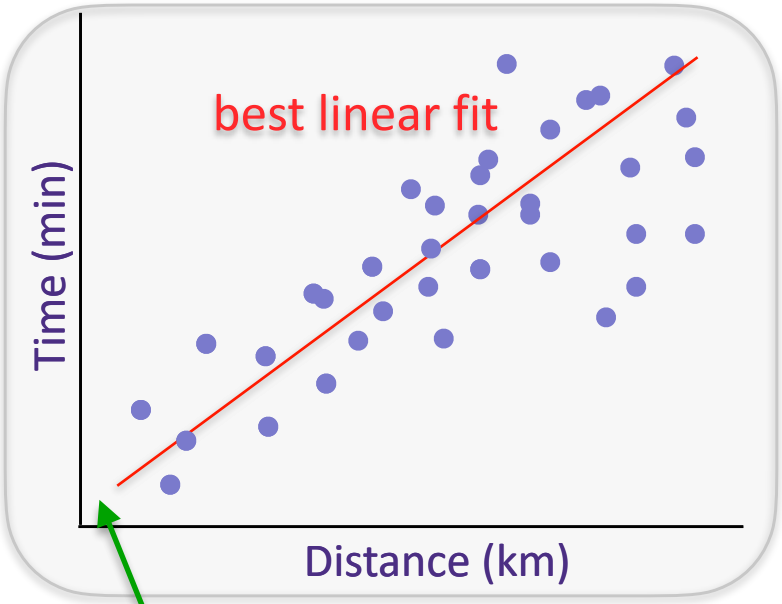
(no intercept for now to make the math easier)

Fit a function to our data, 1-d

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)



Training Data: $\{(x_i, y_i)\}_{i=1}^{n}$ $x_i \in \mathbb{R}$ $y_i \in \mathbb{R}$

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$w \in \mathbb{R}$ *slope of line*

- why the noise?

(no intercept for now to make the math easier)

Process

Decide on a **model**

assume delivery time is a linear function of distance.

Find the function which fits the data best

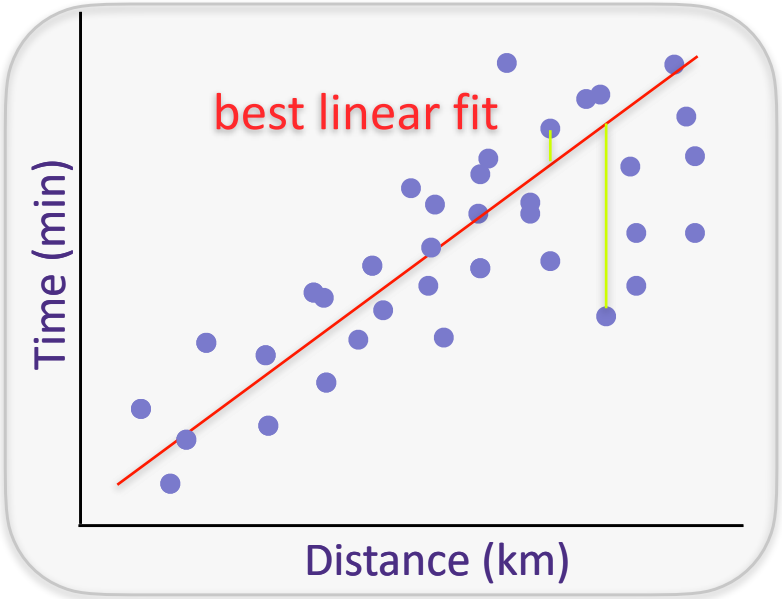
Use function to make prediction on new examples

Fit a function to our data, 1-dimension

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance (km)



Error

$$y_i = x_i w + \epsilon_i$$

Training Data:

$$\{(x_i, y_i)\}_{i=1}^{n}$$

$$x_i \in \mathbb{R}$$
$$y_i \in \mathbb{R}$$

Hypothesis/Model: linear

$$y_i \approx x_i w$$

Loss: least squares solution

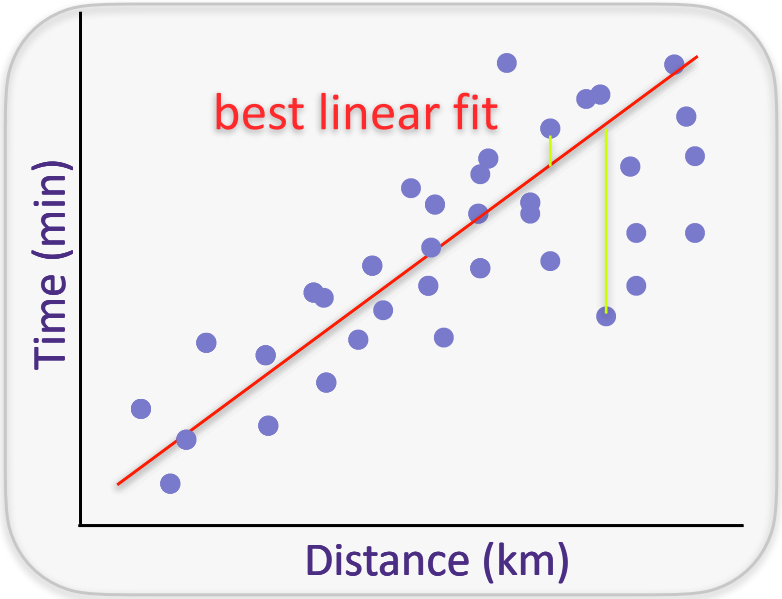
$$\min_w \sum_{i=1}^{n} (y_i - x_i w)^2$$

The regression problem, d-dimensions

Potato delivery logs, predict:

y = delivery time (minutes)

x = distance, traffic, prep time



Error:

$$y_i = x_i w + \epsilon_i$$

Training Data:

$$\{(x_i, y_i)\}_{i=1}^{n} \qquad x_i \in \mathbb{R}^d \quad y_i \in \mathbb{R}$$

Hypothesis/Model: linear

$$y_i \approx x_i^T w$$

Loss: least squares solution

$$\min_w \sum_{i=1}^{n} \left(y_i - x_i^T w\right)^2$$

The regression problem in matrix notation

Data:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

d : # of features

n : # of examples/datapoints

The regression problem in matrix notation

Data:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

d : # of features

n : # of examples/datapoints

Model:

$$\begin{aligned} y_1 &= x_1^T w + \epsilon_1 \\ y_2 &= x_2^T w + \epsilon_2 \\ &\vdots \\ y_n &= x_n^T w + \epsilon_n \end{aligned}$$

$$\mathbf{y} = \mathbf{X}w + \epsilon$$

The regression problem in matrix notation

Data:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

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Model:

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$$\mathbf{y} = \mathbf{X}w + \epsilon$$

Loss: $\widehat{w}_{LS} = \arg\min_w \sum_{i=1}^{n} \left(y_i - x_i^T w\right)^2$

The regression problem in matrix notation

Data:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

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Model:

$$\begin{aligned} y_1 &= x_1^T w + \epsilon_1 \\ y_2 &= x_2^T w + \epsilon_2 \\ &\vdots \\ y_n &= x_n^T w + \epsilon_n \end{aligned}$$

$$\mathbf{y} = \mathbf{X}w + \epsilon$$

Loss: $\widehat{w}_{LS} = \arg\min_w \sum_{i=1}^{n} \left(y_i - x_i^T w\right)^2 = \arg\min_w ||\mathbf{y} - \mathbf{X}w||_2^2$

$$= \arg\min_w (\mathbf{y} - \mathbf{X}w)^T (\mathbf{y} - \mathbf{X}w)$$

The regression problem in matrix notation

$$\begin{aligned}\widehat{w}_{LS} &= \arg\min_{w} ||\mathbf{y} - \mathbf{X}w||_2^2 \\ &= \arg\min_{w} (\mathbf{y} - \mathbf{X}w)^T (\mathbf{y} - \mathbf{X}w)\end{aligned}$$

Set gradient w.r.t. w to zero to find the minima:

The regression problem in matrix notation

$$\begin{aligned}\widehat{w}_{LS} &= \arg\min_{w} ||\mathbf{y} - \mathbf{X}w||_2^2 \\ &= \arg\min_{w} (\mathbf{y} - \mathbf{X}w)^T(\mathbf{y} - \mathbf{X}w) \\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}\end{aligned}$$

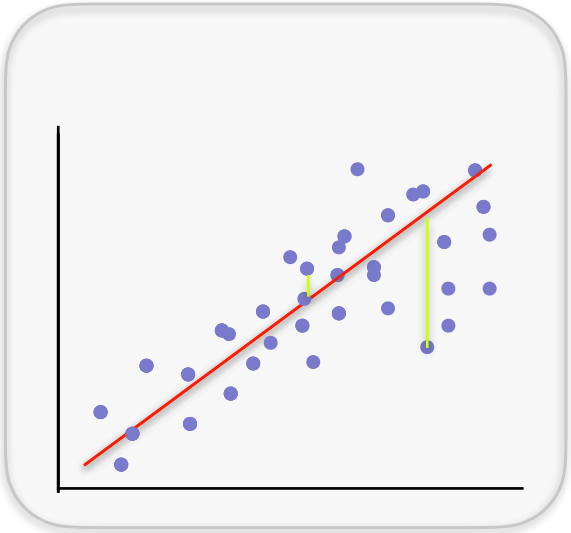
"Closed form" solution!

The regression problem in matrix notation

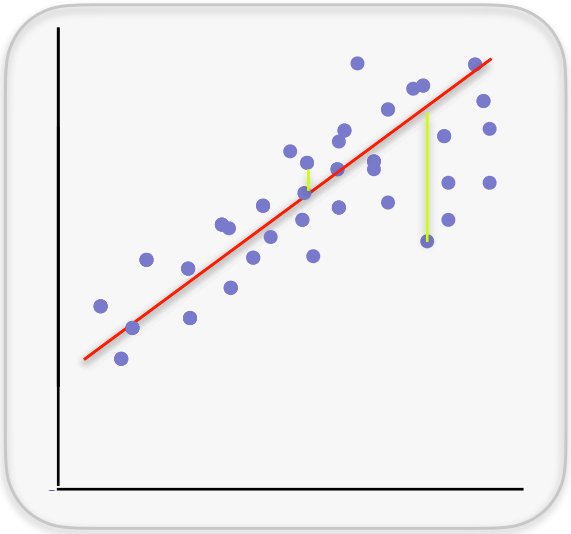
Linear model: $y_i = x_i^T w + \epsilon_i$

Least squares solution:

$$\widehat{w}_{LS} = \arg\min_w ||\mathbf{y} - \mathbf{X}w||_2^2$$
$$= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$



What about an offset (a.k.a intercept)?

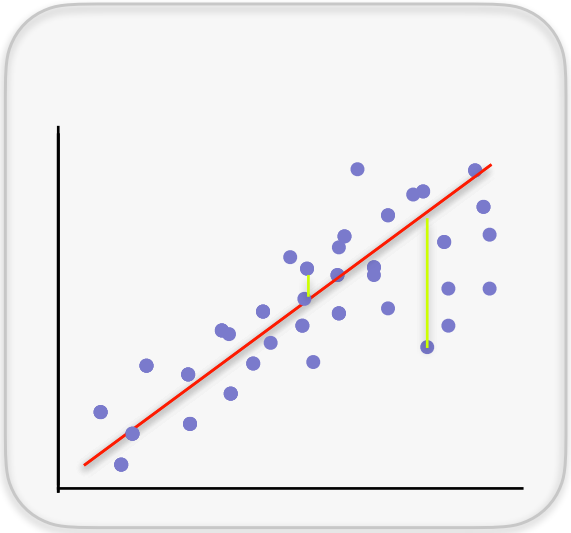


The regression problem in matrix notation

Linear model: $y_i = x_i^T w + \epsilon_i$

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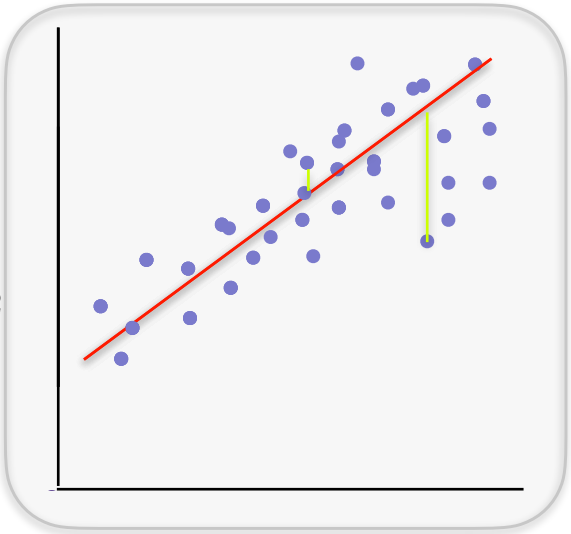
$$\widehat{w}_{LS} = \arg\min_w ||\mathbf{y} - \mathbf{X}w||_2^2$$
$$= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$



Affine model: $y_i = x_i^T w + b + \epsilon_i$

Least squares solution:

$$\widehat{w}_{LS}, \widehat{b}_{LS} = \arg\min_{w,b} \sum_{i=1}^{n} \left(y_i - (x_i^T w + b)\right)^2$$
$$= \arg\min_{w,b} ||\mathbf{y} - (\mathbf{X}w + \mathbf{1}b)||_2^2$$



Dealing with an offset

$$\widehat{w}_{LS}, \widehat{b}_{LS} = \arg\min_{w,b} ||\mathbf{y} - (\mathbf{X}w + \mathbf{1}b)||_2^2$$

Set gradient w.r.t. w and b to zero to find the minima:

Dealing with an offset

$$\widehat{w}_{LS}, \widehat{b}_{LS} = \arg\min_{w,b} ||\mathbf{y} - (\mathbf{X}w + \mathbf{1}b)||_2^2$$

$$\mathbf{X}^T\mathbf{X}\widehat{w}_{LS} + \widehat{b}_{LS}\mathbf{X}^T\mathbf{1} = \mathbf{X}^T\mathbf{y}$$

$$\mathbf{1}^T\mathbf{X}\widehat{w}_{LS} + \widehat{b}_{LS}\mathbf{1}^T\mathbf{1} = \mathbf{1}^T\mathbf{y}$$

If $\mathbf{X}^T\mathbf{1} = 0$, if the features have zero mean,

$$\widehat{w}_{LS} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

$$\widehat{b}_{LS} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

Dealing with an offset

$$\widehat{w}_{LS}, \widehat{b}_{LS} = \arg\min_{w,b} ||\mathbf{y} - (\mathbf{X}w + \mathbf{1}b)||_2^2$$

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If $\mathbf{X}^T\mathbf{1} = 0$,

$$\widehat{w}_{LS} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

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In general, when $\mathbf{X}^T\mathbf{1} \neq 0$,

Dealing with an offset

$$\widehat{w}_{LS}, \widehat{b}_{LS} = \arg\min_{w,b} ||\mathbf{y} - (\mathbf{X}w + \mathbf{1}b)||_2^2$$

$$\mathbf{X}^T\mathbf{X}\widehat{w}_{LS} + \widehat{b}_{LS}\mathbf{X}^T\mathbf{1} = \mathbf{X}^T\mathbf{y}$$
$$\mathbf{1}^T\mathbf{X}\widehat{w}_{LS} + \widehat{b}_{LS}\mathbf{1}^T\mathbf{1} = \mathbf{1}^T\mathbf{y}$$

If $\mathbf{X}^T\mathbf{1} = 0$,

$$\widehat{w}_{LS} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

$$\widehat{b}_{LS} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

In general, when $\mathbf{X}^T\mathbf{1} \neq 0$,

$$\mu = \frac{1}{n}\mathbf{X}^T\mathbf{1}$$

$$\widetilde{\mathbf{X}} = \mathbf{X} - \mathbf{1}\mu^T$$

$$\widehat{w}_{\mathrm{LS}} = (\widetilde{\mathbf{X}}^T\widetilde{\mathbf{X}})^{-1}\widetilde{\mathbf{X}}^T\mathbf{y}$$

$$\widehat{b}_{\mathrm{LS}} = \frac{1}{n}\mathbf{1}^T\mathbf{y} - \mu^T\widehat{w}_{\mathrm{LS}}$$

Process

Decide on a **model:** $y_i = x_i^T w + b + \epsilon_i$

Choose a loss function - least squares

Pick the function which minimizes loss on data

$$\widehat{w}_{LS}, \widehat{b}_{LS} = \arg\min_{w,b} \sum_{i=1}^{n} \left(y_i - (x_i^T w + b)\right)^2$$

Use function to make prediction on new examples

$$\hat{y}_{\text{new}} = x_{\text{new}}^T \hat{w}_{LS} + \hat{b}_{LS}$$

Dealing with an offset - Revisted

$$\widehat{w}_{LS}, \widehat{b}_{LS} = \arg\min_{w,b} ||\mathbf{y} - (\mathbf{X}w + \mathbf{1}b)||_2^2$$

$$\widetilde{\mathbf{X}} = (\mathbf{X}, \mathbf{1})$$

$$\widetilde{\mathbf{X}}\widetilde{w} = \mathbf{X}w + b\mathbf{1} \quad \text{with} \quad \widetilde{w} = (w, b)$$

Why is least squares a good loss function?

$$\begin{aligned}\widehat{w}_{LS} &= \arg\min_{w} ||\mathbf{y} - \mathbf{X}w||_2^2 \\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}\end{aligned}$$

$$P(y_i; x_i, w, \sigma) \; =$$

Why is least squares a good loss function?

$$\widehat{w}_{LS} = \arg\min_{w} ||\mathbf{y} - \mathbf{X}w||_2^2$$
$$= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Consider $y_i = x_i^T w + \epsilon_i$ where $\epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$

$P(y_i; x_i, w, \sigma) =$

Why is least squares a good loss function?

Maximum Likelihood Estimator:

$$\begin{aligned}\widehat{w}_{\mathrm{MLE}} &= \arg\max_{w} \log P(\{y_i\}_{i=1}^n; \{x_i\}_{i=1}^n, w, \sigma) \\ &= \arg\max_{w} -n\log(\sigma\sqrt{2\pi}) + \sum_{i=1}^{n} -\frac{(y_i - x_i^T w)^2}{2\sigma^2}\end{aligned}$$

Why is least squares a good loss function?

Maximum Likelihood Estimator:

$$\begin{aligned}
\widehat{w}_{\text{MLE}} &= \arg\max_{w} \log P(\{y_i\}_{i=1}^n; \{x_i\}_{i=1}^n, w, \sigma) \\
&= \arg\max_{w} -n\log(\sigma\sqrt{2\pi}) + \sum_{i=1}^{n} -\frac{(y_i - x_i^T w)^2}{2\sigma^2} \\
&= \arg\min_{w} \sum_{i=1}^{n} (y_i - x_i^T w)^2
\end{aligned}$$

Recall: $$\widehat{w}_{LS} = \arg\min_{w} \sum_{i=1}^{n} \left(y_i - x_i^T w\right)^2$$

$$\boxed{\widehat{w}_{LS} = \widehat{w}_{MLE} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}}$$

Analysis of Error

$$\text{if} \quad y_i = x_i^T w + \epsilon_i \quad \text{and} \quad \epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \qquad \mathbf{Y} = \mathbf{X}w + \epsilon$$

$$\begin{aligned} \widehat{w}_{MLE} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T(\mathbf{X}w + \epsilon) \\ &= w + (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\epsilon \end{aligned}$$

Maximum Likelihood Estimator is unbiased:

Analysis of Error

$$\text{if} \quad y_i = x_i^T w + \epsilon_i \quad \text{and} \quad \epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \qquad \mathbf{Y} = \mathbf{X}w + \epsilon$$

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Covariance is:

Analysis of Error

$$\text{if} \quad y_i = x_i^T w + \epsilon_i \quad \text{and} \quad \epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \qquad \mathbf{Y} = \mathbf{X}w + \epsilon$$

$$\begin{aligned} \widehat{w}_{MLE} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T(\mathbf{X}w + \epsilon) \\ &= w + (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\epsilon \end{aligned}$$

$$\mathbb{E}[\widehat{w}_{MLE}] = w$$

$$\text{Cov}(\widehat{w}_{MLE}) = \mathbb{E}[(\widehat{w} - \mathbb{E}[\widehat{w}])(\widehat{w} - \mathbb{E}[\widehat{w}])^T] = (\mathbf{X}^T\mathbf{X})^{-1}$$

$$\widehat{w}_{MLE} \sim \mathcal{N}(w, (\mathbf{X}^T\mathbf{X})^{-1})$$