CSE 589
Applied Algorithms
Spring 1999

Huffman Coding
Arithmetic Coding

Huffman Coding

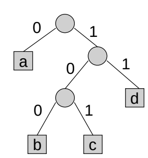
- Huffman (1951)
- Uses frequencies of symbols in a string to build a variable rate prefix code.
  - Each symbol is mapped to a binary string.
  - More frequent symbols have shorter codes.
  - No code is a prefix of another.
- Example: a 0
  b 100
  c 101
  d 11

Variable Rate Code Example

- Example: a 0, b 100, c 101, d 11
- Coding:
  - aabddcaa = 16 bits
  - 0 0 100 11 11 101 0 0= 14 bits
- Prefix code ensures unique decodability.
  - 00100111110100
  - a a b d d c a a
- Morse Code an example of variable rate code. E = . and Z = _ _ . .

Huffman Tree for a Prefix Code

- Example: a 0, b 100, c 101, d 11

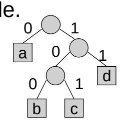


Leaves are labeled with symbols.

The code of a symbol is the sequence of labels on the path from the root to the symbol.

Encoding and Decoding

- Encoding: send the code, then the encoded data
  - x = aabddcaa
  - c(x) = $\underbrace{\text{a0,b100,c101,d11,}}_{\text{Huffman code}}\underbrace{\text{0010011111010 0}}_{\text{code of x}}$
- Decoding: Build the Huffman tree, then use it to decode.




```
repeat
start at root of tree
  repeat
  if node is not a leaf
    if read bit = 1 then go right
    else go left
  report leaf
```


Cost of a Huffman Tree

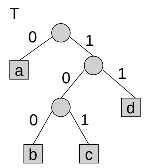
- Let $p_1, p_2, \ldots, p_m$ be the probabilities for the symbols $a_1, a_2, \ldots, a_m$, respectively.
- Define the cost of the Huffman tree T to be

$$C(T) = \sum_{i=1}^{m} p_i r_i$$

where r_i is the length of the path from the root to a_i.
- $C(T)$ is the expected length of the code of a symbol coded by the tree T.

Example of Cost

- Example: a 1/2, b 1/8, c 1/8, d 1/4



C(T) = 1 x 1/2 + 3 x 1/8 + 3 x 1/8 + 2 x 1/4 = 1.75
a b c d

Optimal Huffman Tree

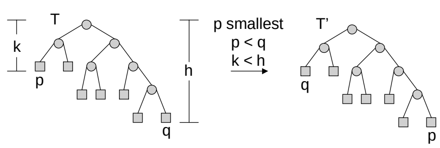
- Input: Probabilities $p_1, p_2, \ldots, p_m$ for symbols $a_1, a_2, \ldots, a_m$, respectively.
- Output: A Huffman tree that minimizes the average number of bits to code a symbol. That is, minimizes

$$C(T) = \sum_{i=1}^{m} p_i r_i$$

where r_i is the length of the path from the root to a_i.

Optimality Principle 1

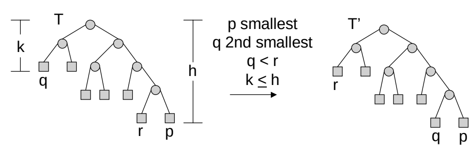
- In an optimal Huffman tree a lowest probability symbol has maximum distance from the root.
  - If not exchanging a lowest probability symbol with one at maximum distance will lower the cost.



C(T') = C(T) + hp - hq + kq - kp = C(T) - (h-k)(q-p) < C(T)

Optimality Principle 2

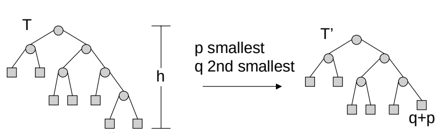
- The second lowest probability is a sibling of the the smallest in some optimal Huffman tree.
  - If not, we can move it there not raising the cost.



C(T') = C(T) + hq - hr + kr - kq = C(T) - (h-k)(r-q) ≤ C(T)

Optimality Principle 3

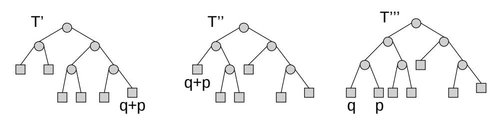
- Assuming we have an optimal Huffman tree T whose two lowest probability symbols are siblings at maximum depth, they can be replaced by a new symbol whose probability is the sum of their probabilities.
  - The resulting tree is optimal for the new symbol set.



C(T') = C(T) + (h-1)(p+q) - hp -hq = C(T) - (p+q)

Optimality Principle 3 (cont')

- If T' were not optimal then we could find a lower cost tree T''. This will lead to a lower cost tree T''' for the original alphabet.



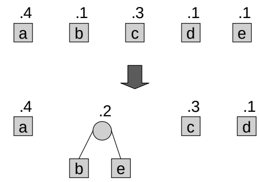
C(T''') = C(T'') + p + q < C(T') + p + q = C(T) which is a contradiction

Huffman Tree Algorithm

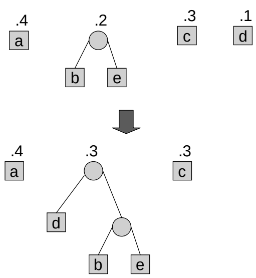
```
form a node for each symbol a_i with weight p_i;
insert the nodes in a min priority queue ordered by probability;
while the priority queue has more than one element do
   min1 := delete-min;
   min2 := delete-min;
   create a new node n;
   n.weight := min1.weight + min2.weight;
   n.left := min1;
   n.right := min2;
   insert(n)
return the last node in the priority queue.
```


Example of Huffman Tree Algorithm (1)

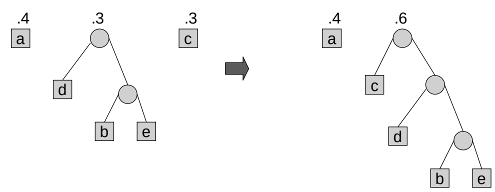
- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1



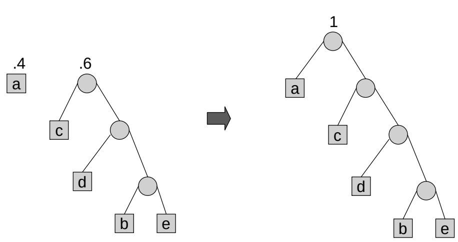
Example of Huffman Tree Algorithm (2)



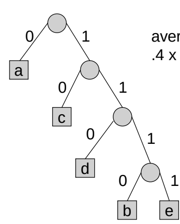
Example of Huffman Tree Algorithm (3)



Example of Huffman Tree Algorithm (4)



Optimal Huffman Code



average number of bits per symbol is
.4 x 1 + .1 x 4 + .3 x 2 + .1 x 3 + .1 x 4 = 2.1

a	0
b	1110
c	10
d	110
e	1111

Huffman Code vs. Entropy

- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1

Entropy

H = -(.4 x $\log_2$(.4) + .1 x $\log_2$(.1) + .3 x $\log_2$(.3)
+ .1 x $\log_2$(.1) + .1 x $\log_2$(.1))
= 2.05 bits per symbol

Huffman Code

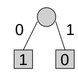
HC = .4 x 1 + .1 x 4 + .3 x 2 + .1 x 3 + .1 x 4
= 2.1 bits per symbol
pretty good!

Quality of the Huffman Code

- The Huffman code is within one bit of the entropy lower bound.

$$H \le HC \le H + 1$$

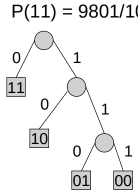
- Huffman code does not work well with a two symbol alphabet.
  - Example: P(0) = 1/100, P(1) = 99/100
  - HC = 1 bits/symbol



  - H = -((1/100)*$\log_2$(1/100) + (99/100)$\log_2$(99/100))
    = .08 bits/symbol

Extending the Alphabet

- Assuming independence P(ab) = P(a)P(b), so we can lump symbols together.
- Example: P(0) = 1/100, P(1) = 99/100
  - P(00) = 1/10000, P(01) = P(10) = 99/10000,
    P(11) = 9801/10000.



HC = 1.03 bits/symbol (2 bit symbol)
= .515 bits/bit

Still not that close to H = .08 bits/bit

Including Context

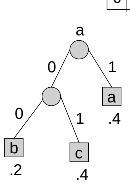
- Suppose we add a one symbol context. That is in compressing a string $x_1 x_2 ... x_n$ we want to take into account x_{k-1} when encoding x_k.
  - New model, so entropy based on just independent probabilities of the symbols doesn't hold. The new entropy model (2nd order entropy) has for each symbol a probability for each other symbol following it.
  - Example: {a,b,c}

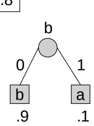
prev \ next	a	b	c
a	.4	.2	.4
b	.1	.9	0
c	.1	.1	.8

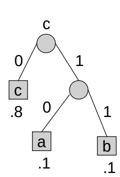
Multiple Codes

prev \ next	a	b	c
a	.4	.2	.4
b	.1	.9	0
c	.1	.1	.8

Code for first symbol
a 00
b 01
c 10







a b b a c c

00 00 0 1 01 0

Notes on Huffman Codes

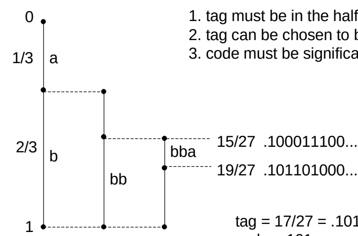
- Time to design Huffman Code is O(n log n) where n is the number of symbols.
- Typically, for compressing a string the probabilities are chosen as the actual frequencies of the symbols in the string.
- Huffman works better for larger alphabets.
- There are adaptive (one pass) Huffman coding algorithms. No need to transmit code.
- Huffman is still popular. It is simple and it works.

Arithmetic Coding

- Huffman coding works well for larger alphabets and gets to within one bit of the entropy lower bound. Can we do better. Yes
- Basic idea in arithmetic coding:
  – represent each string x of length n by an interval A in [0,1).
  – The width of the interval A represents the probability of x occurring.
  – The interval A can itself be represented by any number, called a tag, within the half open interval.
  – The significant bits of the tag is the code of x.

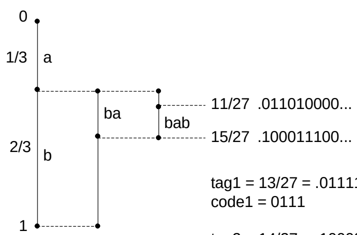
Example of Arithmetic Coding (1)

1. tag must be in the half open interval.
2. tag can be chosen to be (l+r)/2.
3. code must be significant bits of tag.



tag = 17/27 = .101000010...
code = 101

Example of Arithmetic Coding (2)



tag1 = 13/27 = .011110110...
code1 = 0111

tag2 = 14/27 = .100001001...
code2 = 1