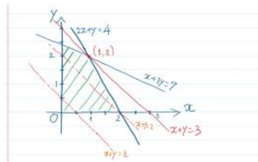
CSEP 521: Linear Programming

Richard Anderson, March 11, 2021



Announcements

- Readings
  - Skim textbook chapters on Linear Programming
    - DasGupta, Papadimitriou, and Vazarani
- Course evaluations
  - You should have received a link on your UW account
- Last homework is due tonight
  - Notify instructor if any homework is going to be turned in after March 14

Linear Programming

A linear program is a problem with n variables $x_1,\ldots,x_n$, that has:

1. A linear objective function, which must be maximized (or minimized).
   $\max c_1x_1+c_2x_2+\ldots+c_nx_n$
2. A set of m linear constraints.
   $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i$ (or $\geq$ or $=$)

Note: the values of the coefficients c_i, $a_{i,j}$ are given in the problem input.

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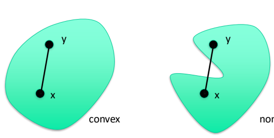
Geometry

- Convexity
- Intersection of convex sets is convex
- Max / Min always at the corner

Convexity

A set of points is *convex* if for all pairs x, y in the set, every point on the line segment $\{ t x+(1-t) y \mid 0 < t < 1 \}$ connecting them is also in the set



convex non-convex

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Convexity and half-spaces

- An inequality
  $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b$
  defines a *half-space*.
- Half-spaces are convex
- Intersections of convex sets are convex
- So, the feasible region for a linear program is always a convex polyhedron

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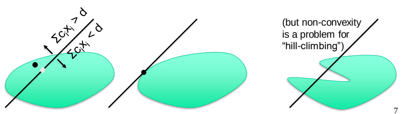
Max/Min is always at a "corner"

- Linear extrema are not in the interior of a convex set
  - E.g.: maximize $c_1x_1+c_2x_2+\ldots+c_nx_n$
  - If max were in the interior, there's always a better interior point just off the hyperplane $c^Tx = d$
- On a polyhedron, max may = line, face, …, but includes vertices thereof, so always a "corner,"
  (though maybe not uniquely a corner)



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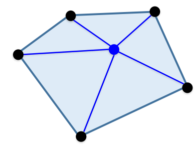
Convex combinations

- Defn: a *convex combination* of points/vectors $p_1,p_2, \ldots, p_n$ is a point $\alpha_1p_1 + \alpha_2p_2 + \ldots + \alpha_2p_n$ where $\alpha_i > 0$ and $\sum_i \alpha_i = 1$
- Fact: the set of all convex combinations of $p_1,p_2, \ldots, p_n$ sweep out their convex hull



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The diet problem

Maximize protein consumption subject to

≤ 5 units of fat per day

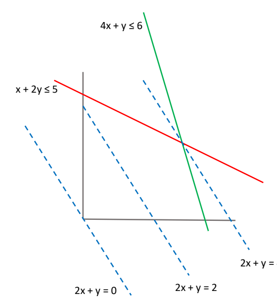
≤ 6 dollars per day

	Protein / lb	Fat / lb	$ / lb
Steak	2	1	4
Peanut Butter	1	2	1

The diet problem

Maximize:
$2x + y$

Subject to:
$4x + y \le 6$
$x + 2y \le 5$
$x \ge 0$
$y \ge 0$



Duality - upper bounds

Maximize:
$2x + y$

Subject to:
$4x + y \le 6$ (1)
$x + 2y \le 5$ (2)
$x \ge 0$
$y \ge 0$

(1):	$2x + y \le 4x + y \le 6$
2(2):	$2x + y \le 2(x + 2y) \le 10$
.5(1) + .25(2)	$2x + y \le .5(4x + y) + .25(x + 2y) \le 3 + 1.25$

What is the best upper bound we can get?

Upper bounds

$a(4x + y) \le 6a$
$b(x + 2y) \le 5b$

If $a(4x + y) + b(x + 2y) \ge 2x + y$
then $6a + 5b \ge \text{opt}$

min $6a + 5b$

Subject to
$4a + b \ge 2$
$a + 2b \ge 1$
$a \ge 0$
$b \ge 0$

A Central Result of LP Theory: Duality Theorem

- Every linear program has a dual
- If the original is a maximization, the dual is a minimization and vice versa
- Solution of one leads to solution of other

Primal: Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge 0$

Dual: Minimize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \ge \mathbf{c}$, $\mathbf{y} \ge 0$

If one has optimal solution so does the other, and their values are the same.

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Duality Theorem

Practical Use of Duality:

- Sometimes LP algorithms will run faster on the dual than on the primal.
- Can be used to bound how far you are from optimal solution.
- Important implications for economists.

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Standard Form of a Linear Program.

Minimize $b_1y_1 + b_2y_2 + \ldots + b_my_m$

subject to $\Sigma_{1 \le i \le m}\, a_{ij}y_i \ge c_j \qquad j = 1..n$

$y_i \ge 0 \qquad i = 1..m$

or

Maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to $\Sigma_{1 \le j \le n}\, a_{ij}x_j \le b_i \qquad i = 1..m$

$x_j \ge 0 \qquad j = 1..n$

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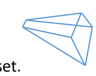
The Feasible Set

- Intersection of a set of half-spaces is called a *polyhedron*.
- A bounded, nonempty polyhedron is a *polytope*.

There are 3 cases:



- feasible set is empty.
- cost function is unbounded on feasible set.
- cost has a minimum (or max) on feasible set.

(First two cases uncommon for real problems in economics and engineering.)

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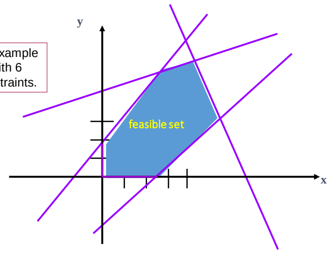
Optimal vector occurs at some corner of the feasible set

An example with 6 constraints.



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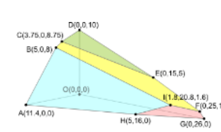
Linear Programming in n dimensions

- Feasible region is an n-dimensional polytope
- Solving the linear program is finding the vertex that maximizes the objective function
- Each vertex is determined by a set of n-constraints (hyperplanes)
- General form n variables, $m \ge n$ constraints
  - Vertices determined by making n constraints tight



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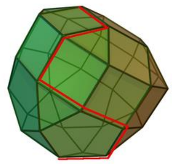
Simplex Algorithm



- Traverse the polytope from vertex to vertex in a direction that increases the objective function
- When a maximum is reached the problem is solved
- Traversing the edges means changing the constraints that are tight

The Simplex Method – more details

Phase 1 (start-up): Find initial feasible vertex.

Phase 2 (iterate):

1. Can the current objective value be improved by swapping a basic variable? If not - stop.
2. Select entering variable, e.g. via greedy heuristic: choose the variable that gives the fastest rate of increase in the objective function value.
3. Select the leaving basic variable by applying the minimum ratio (tightest constraint) test
4. Update equations to reflect new basic feasible solution.

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Degeneracy, Cycling, Pivot Rules

- Some (of many) *pivot rules*:
  - Largest coefficient (Danzig)
  - Largest increase
  - Bland's rule (entering/exiting vars w/ min index)
  - Random edge
  - Steepest edge – seems to be best in practice
- In n dimensions, n hyperplanes can define a point. If more intersect at a vertex, the LP is *degenerate*, and most of the above rules may *stall* there, i.e., "move" to same vertex (with different basis); some may *cycle* there: infinite loop
- Pathological cases for simplex algorithm can take an exponential number of steps
  - Not a problem in practice, but this bugged the theorists for years

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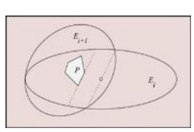
Simplex Algorithm Runtime

- Algorithm implemented by (m+1) X (m + n + 1) array with row and column operations
- Performs well in practice
- Worst case performance is exponential on the Klee-Minty cube
- Randomized versions are polynomial time
- Average case is polynomial time

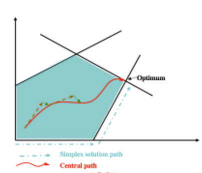
Other algorithms for Linear Programming

- Exterior methods
  - Bound the polytope to find the maximum
  - Khachiyan [1979]
  - First polynomial time algorithm
    - But not really practical
    - And has numerical stability issues



Interior Point Methods

- Instead of working on the surface of the polytope (as the simplex algorithm) go through the interior of the polytope
- Karmarkar [1987] developed a poly time (and practical) method
- Key idea is to have a path in the inside of the polytope that does not break through
- Interior point methods have runtimes of $O(n^3)$ (or less)



Linear programming algorithms

$$
\begin{aligned}
\text{maximize} \quad & z = x + 2y \\
\text{subject to:} \quad & 2x + y \le 20 \\
& -4x + 5y \le 10 \\
& -x + 2y \ge -2 \\
& -x + 5y = 15 \\
& x \ge 0 \\
& y \ge 0
\end{aligned}
$$

→ LP Solver →

Linear Program for the Assignment Problem

- Weighted bipartite matching
- Choose assignments to maximize profit

.32	.41	.95	.83	.11
.81	.22	.54	.91	.61
.02	.41	.24	.76	.86
.16	.85	.52	.85	.02
.76	.82	.82	.59	.65

$0 \le x_{ij} \le 1$

$\sum_i x_{ij} = 1$

$\sum_j x_{ij} = 1$

$\max \sum_{ij} a_{ij} x_{ij}$

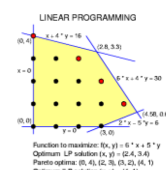
$x_{ij} = 1$, edge (i,j) is matched
$x_{ij} = 0$, edge (i,j) is unmatched

Fractional solutions

- Optimal solution may be fractional: $x_{12} = 0.5$ and $x_{13} = 0.5$
- For the assignment problem (and network flow), there is a theorem which guarantees the existence of an integer solution
  - ``Total Unimodularity''
- Specific algorithms (e.g., Hungarian Algorithm) will generate the integral solution
- LP Algorithm may support the integral solution
- Post processing of LP solution may also find integer solution
  - Round the values while preserving the optimal value

Integer Linear Programming

- What if we have a linear program where we require (some) variables to have integer solutions
- Rounding does NOT always work
  - Fractional solutions can be a long way from integer solutions
  - NP Complete problems can be reduced to ILPs with 0-1 variables
  - Reduction from satisfiability



Linear Programming Summary

- High dimensional algorithms with practical solutions
- Relies on geometry
- Bridge between Applied Math and Computer Science

Course summary

- What was this course about?
- Algorithmic ideas
  - Randomness and average case
  - Hashing – the use of random functions
  - Stream based algorithms – small space computation
  - Working in high dimensions


"That's all Folks!"