CSEP 521: Network Flow and Linear Programming

Richard Anderson,   March 9, 2021

Announcements

- Remaining lectures on Optimization
  - Combinatorial Optimization
  - Linear Programming

- Course Evaluation
  - You will have received a link

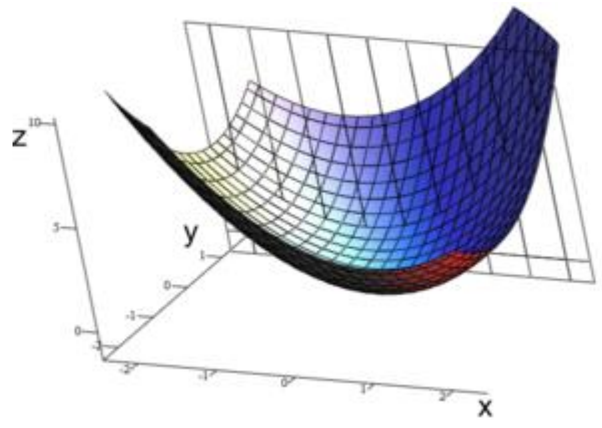
- Readings
  - Skim textbook chapters on Matching/Network Flow (CLRS or KT)
  - Skim textbook chapters on Linear Programming
    - DasGupta, Papadimitriou, and Vazarani

- Last homework is due Thursday, March 11
  - Notify instructor if any homework is going to be turned in after March 14

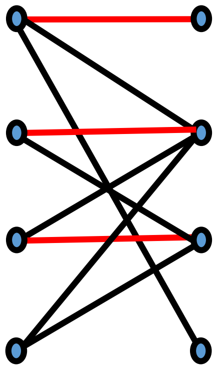
Optimization

- Solve a problem by expressing it as minimizing or maximizing a real valued function
- Examples:
  - Page layout
  - Allocation of industrial materials for a five year plan
  - Placement of on-line ads
  - Pricing of airline seats

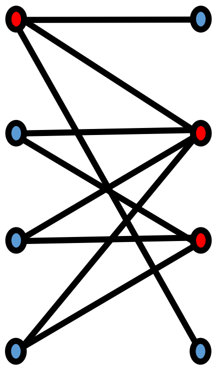
Optimization



- Local improvement algorithms
  - Iteratively improve solution until a local maximum (or minimum) is reached
  - Prove that the maximum is a global maximum
- Duality
  - Pairs of problems that bound solutions
  - Finding the maximum for one problem finds the minimum for another problem

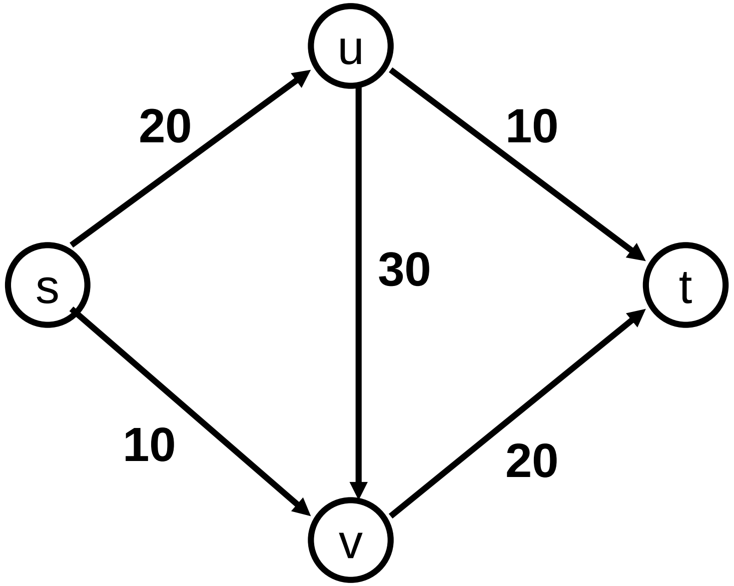


Bipartite Matching



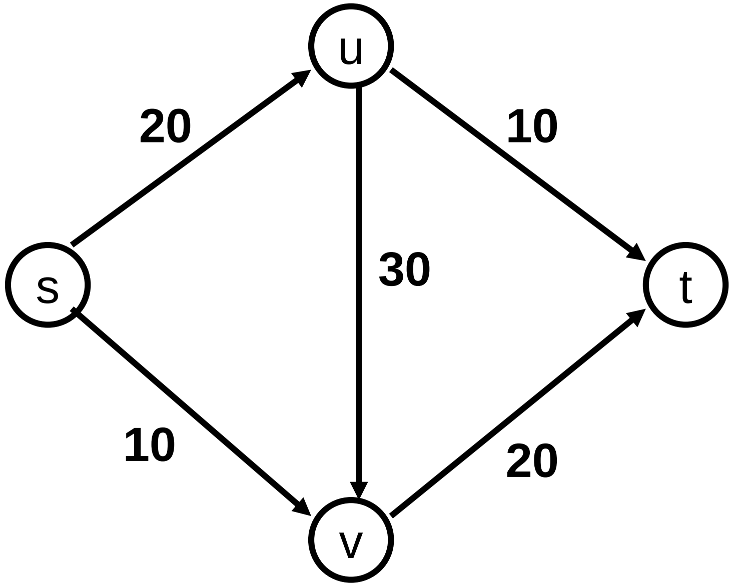
Vertex Cover

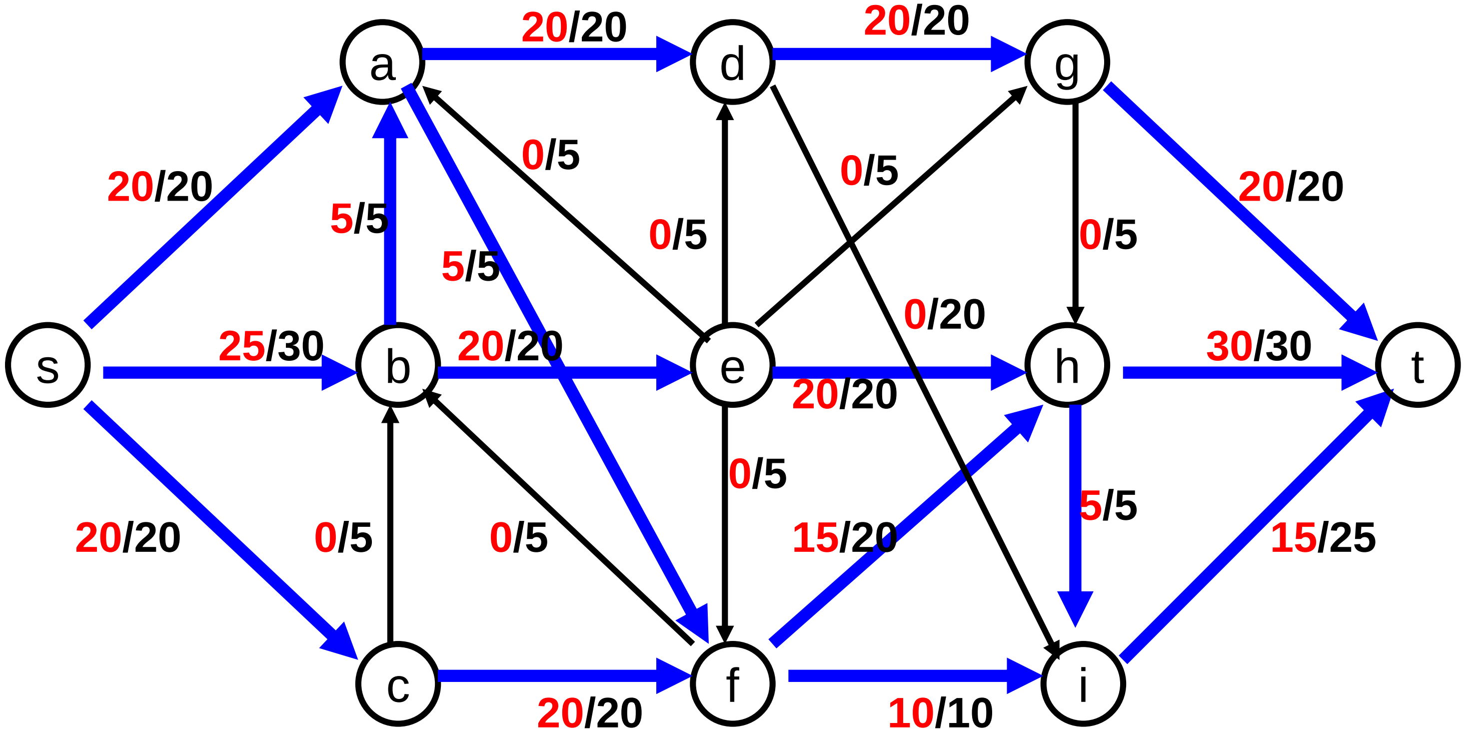
Network Flow



Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible





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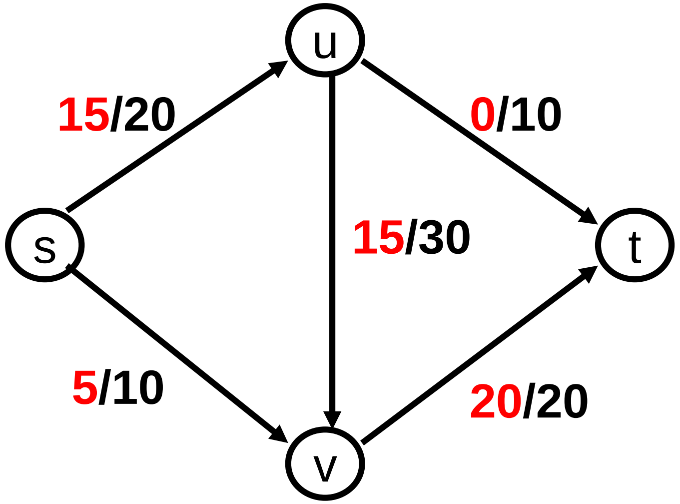
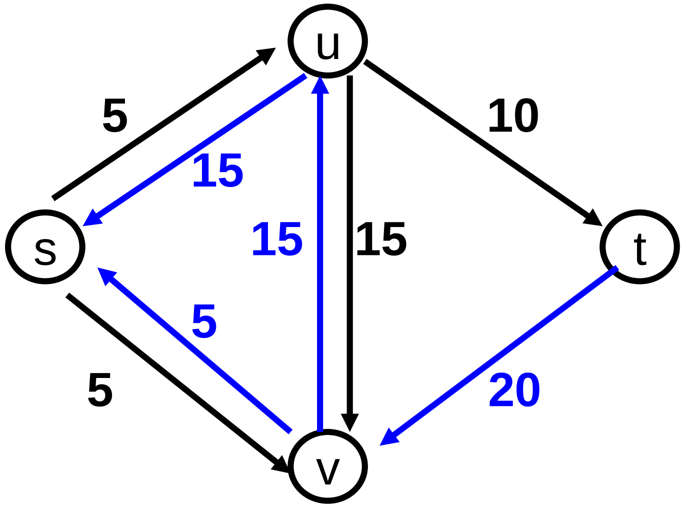
Residual Graph

- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f



- Find a path from s to t in G_R with minimum edge capacity (in G_R) of b > 0
- Add flow b to the path from s to t in G



Ford-Fulkerson Algorithm (1956)

while not done

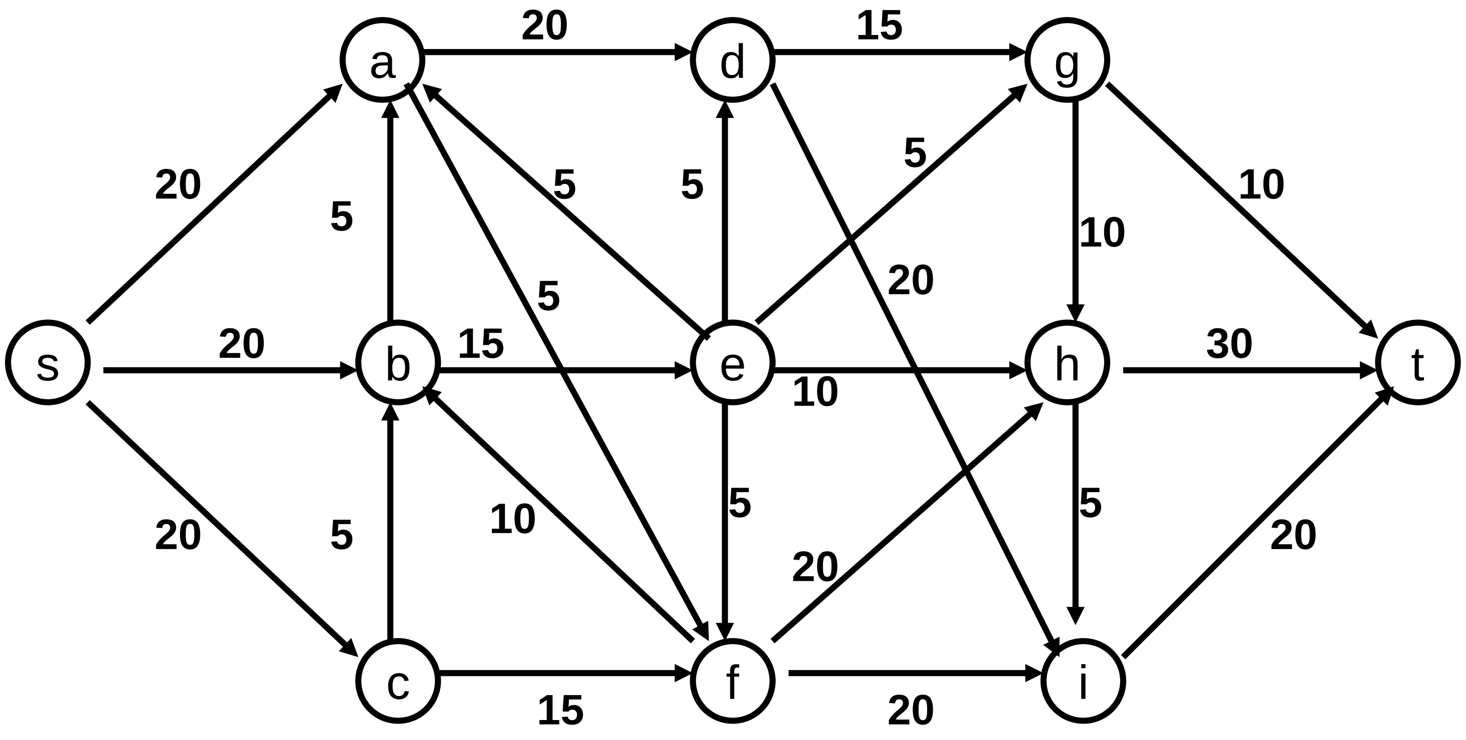
Construct residual graph G_R

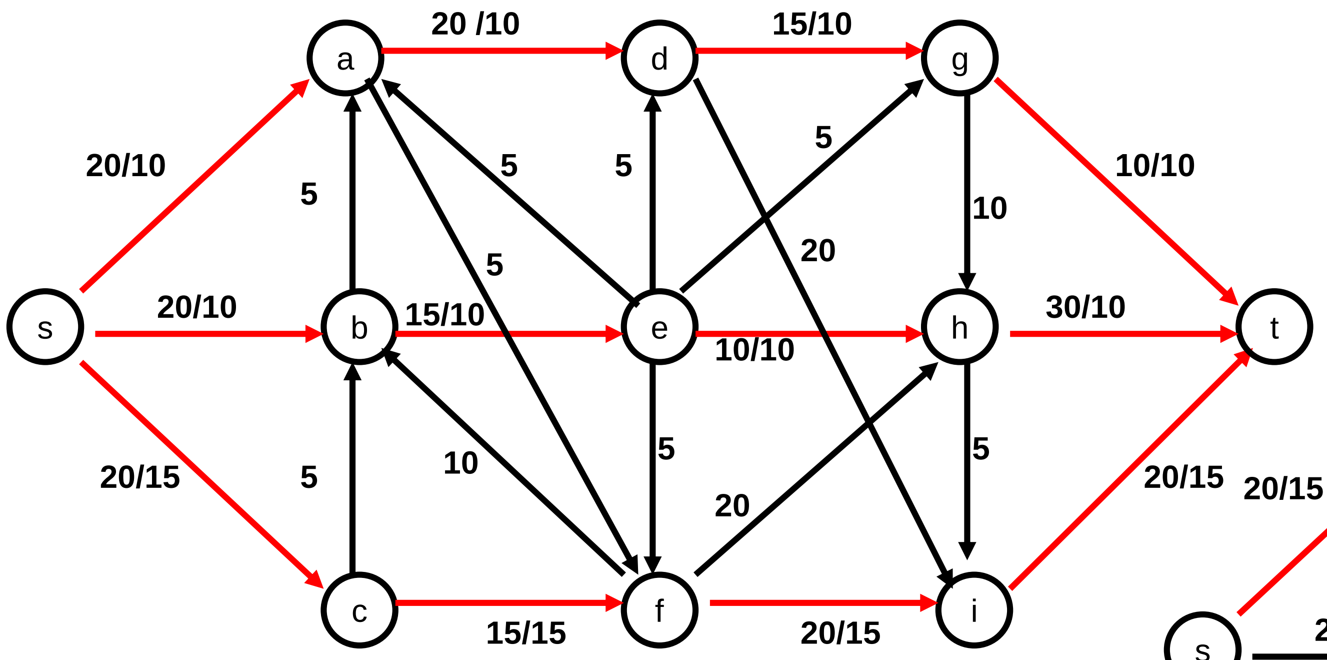
Find an s-t path P in G_R with capacity $b > 0$

Add b units along P in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

Flow Example

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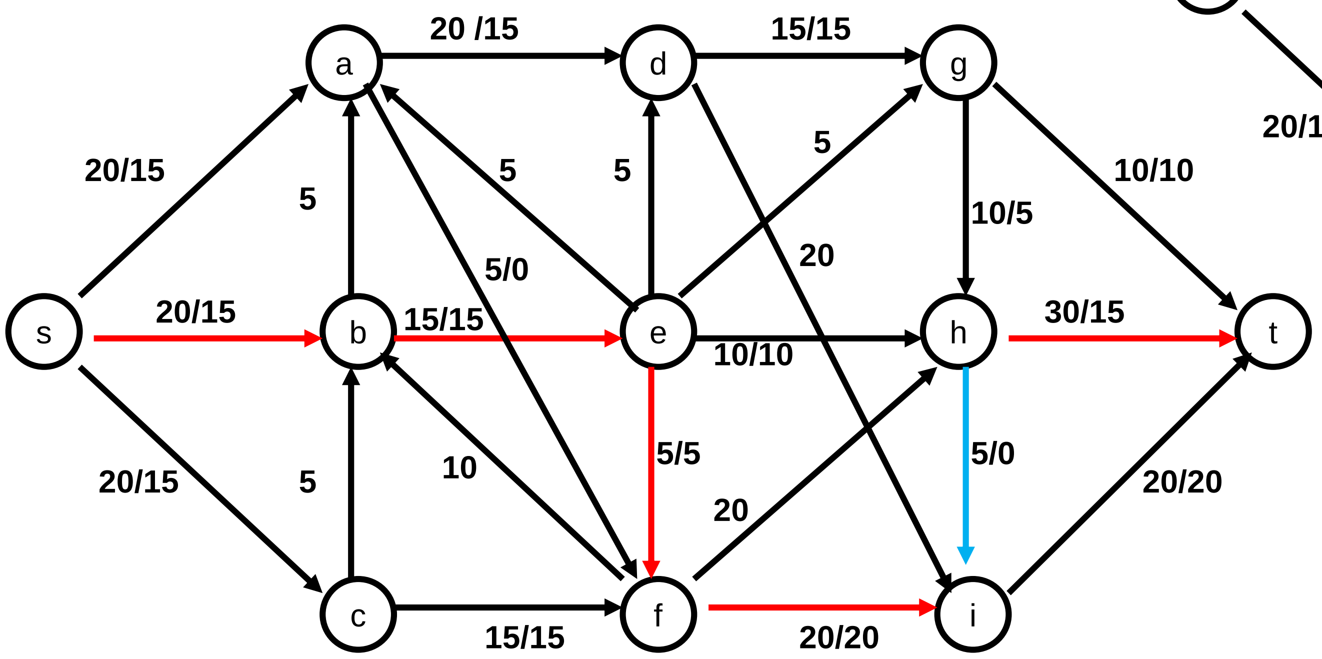
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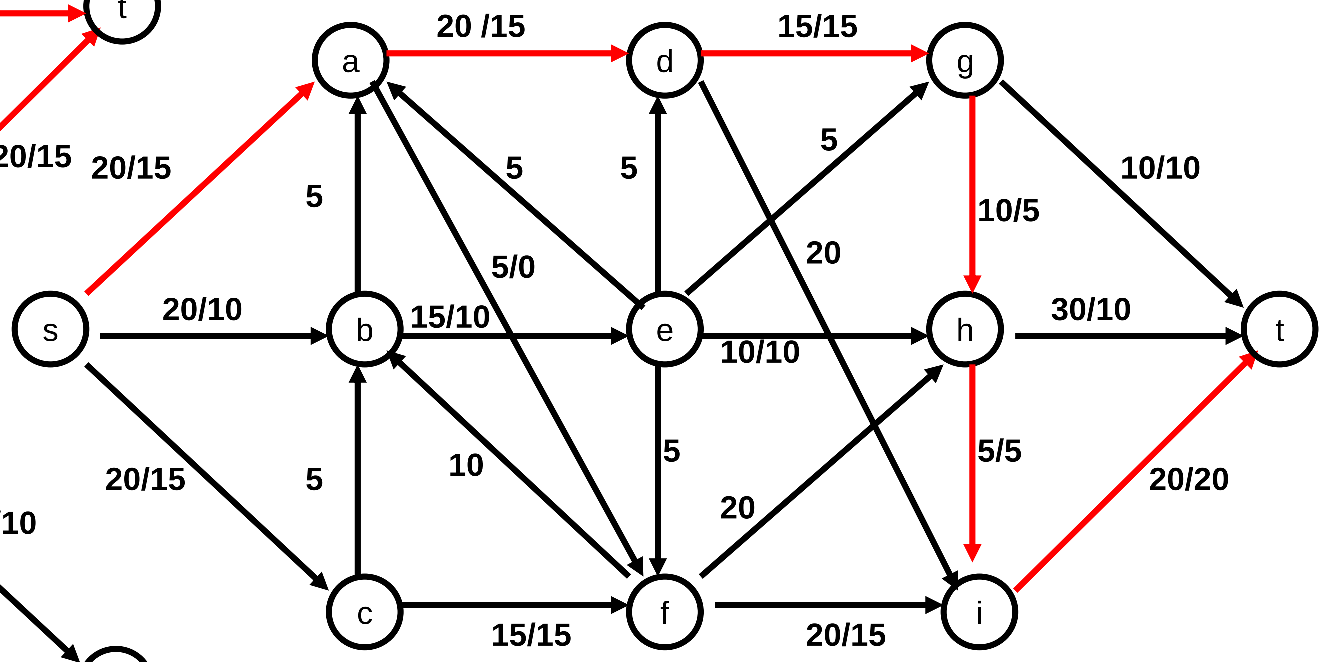
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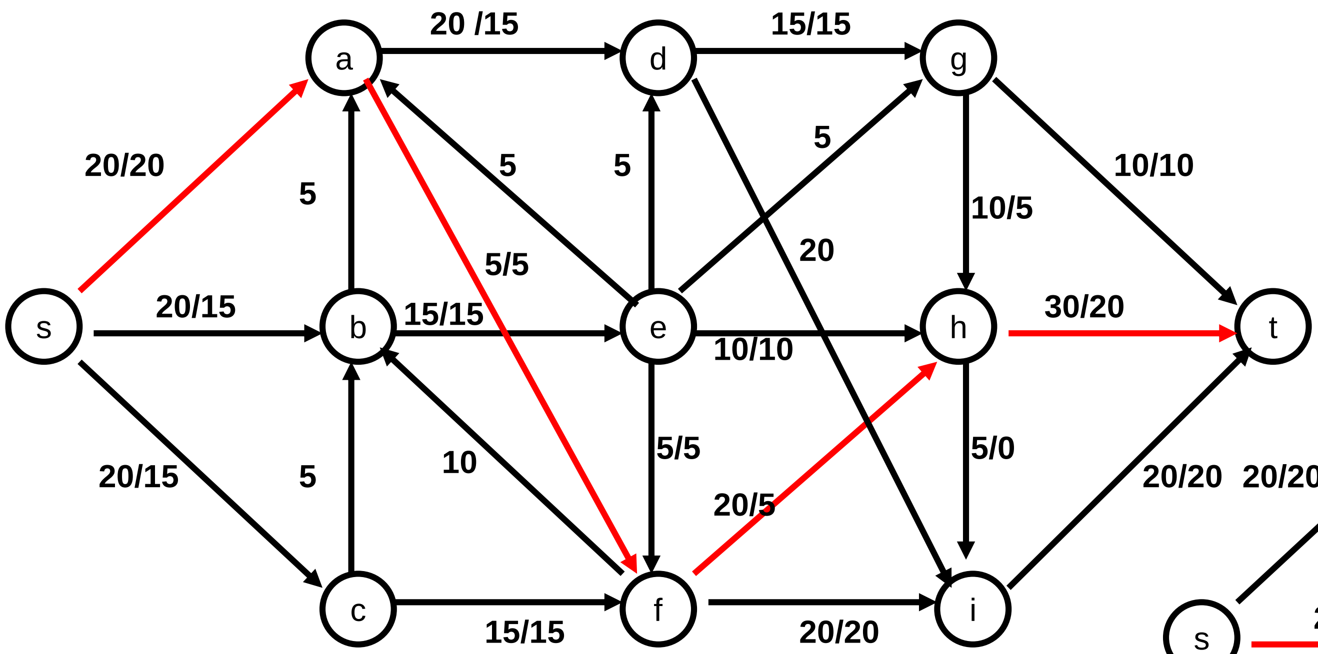

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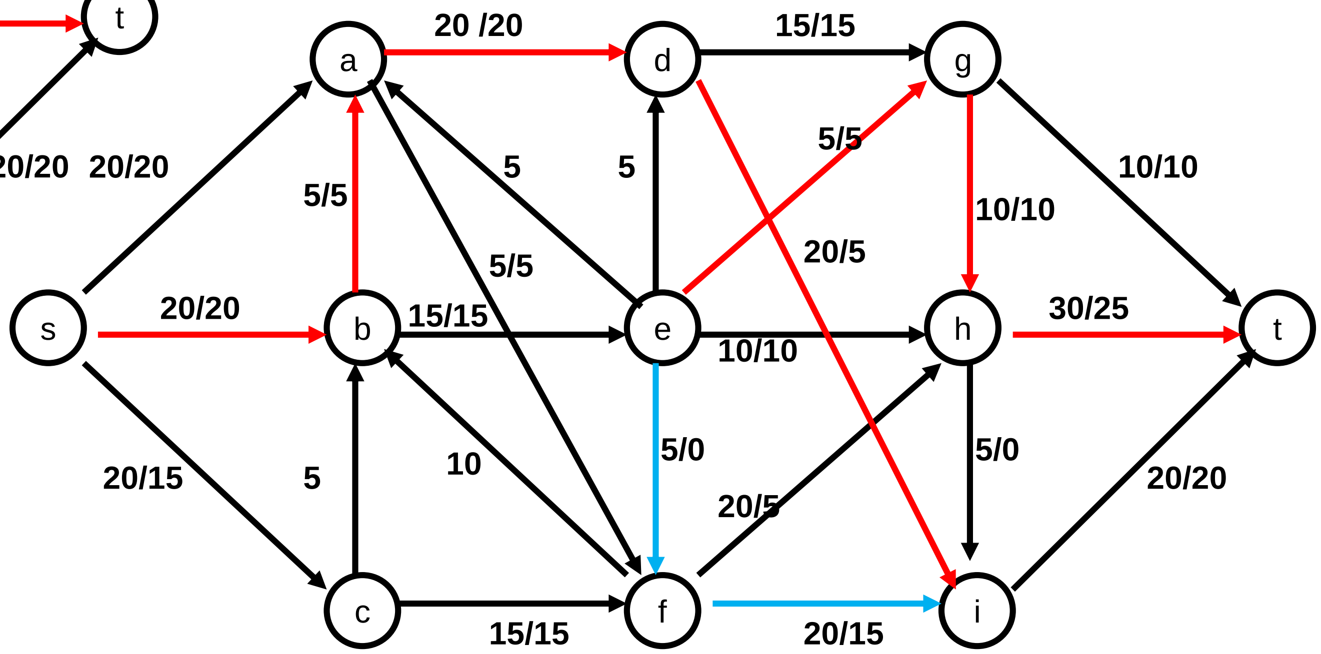

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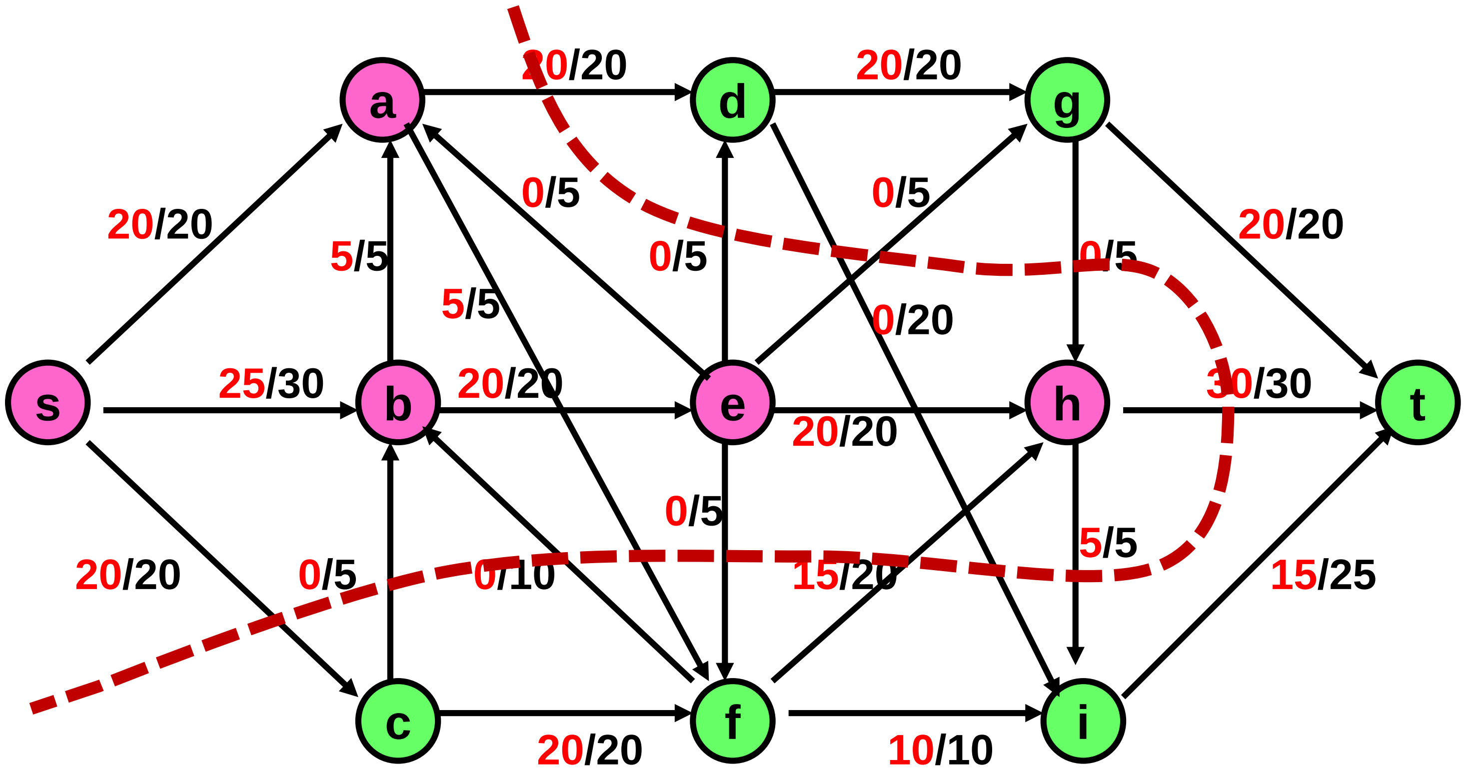
Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S

- Flow(S,T) <= Cap(S,T)

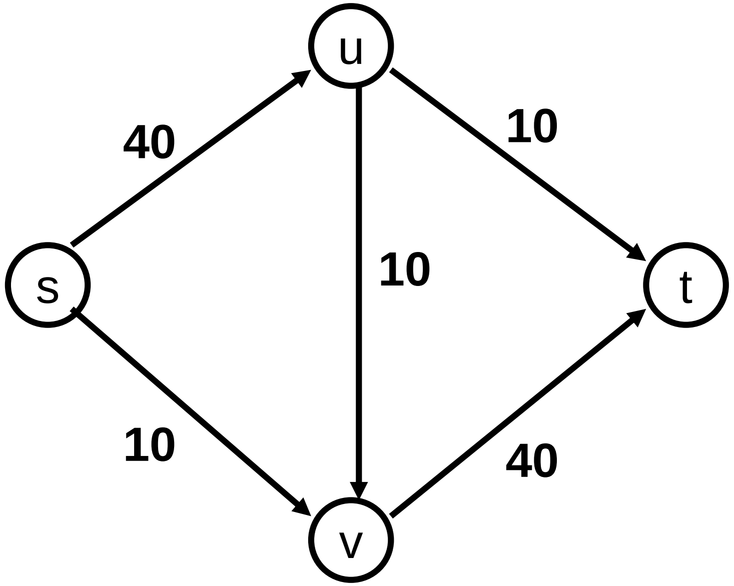
Cap(S,T) and Flow(S,T)

S={s, a, b, e, h},   T = {c, f, i, d, g, t}



Cap(S,T) = 95,   Flow(S,T) = 80 – 15 = 65

Minimum value cut

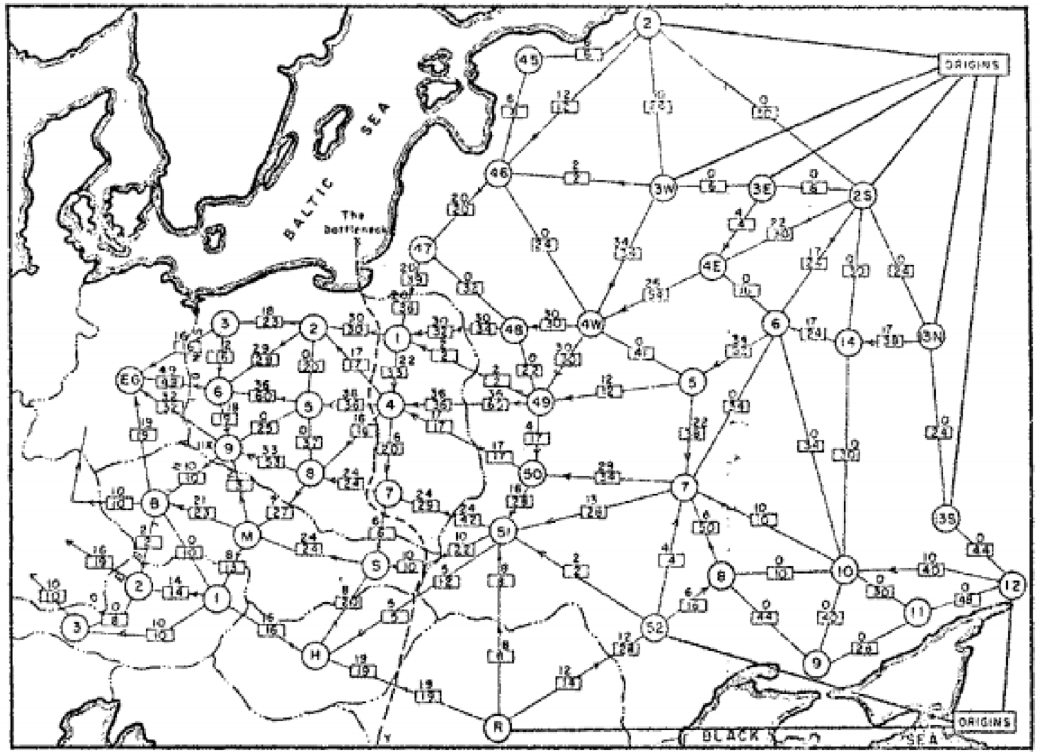


MaxFlow – MinCut Theorem

- There exists a flow which has the same value as the minimum cut
- This shows that both the flow is maximum and the cut is minimum since the flow is always less than or equal to the cut
- The proof is to run the Ford-Fulkerson algorithm until it completes and look at the residual graph
- This is a ``duality theorem'' as the MaxFlow and MinCut problems are duals

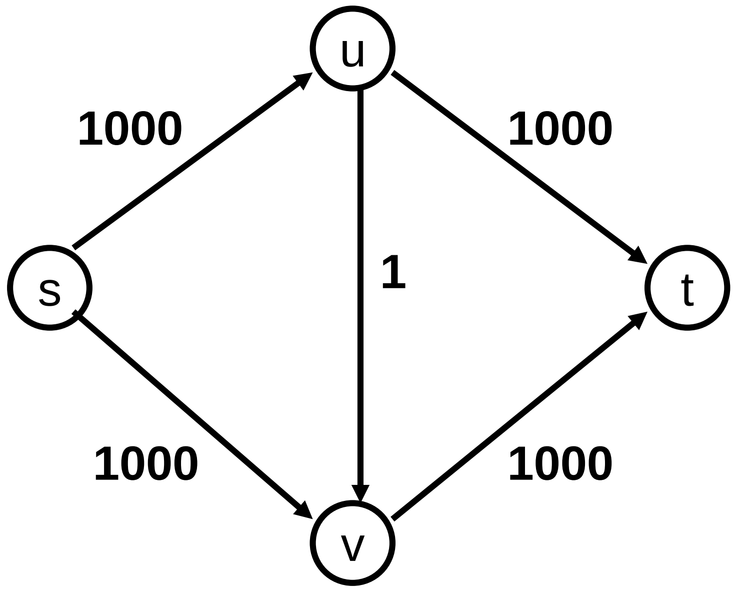
History

- Ford / Fulkerson studied network flow in the context of the Soviet Rail Network



Ford Fulkerson Runtime

- Cost per phase times number of phases
- Phases
  - Capacity leaving source: C
  - Add at least one unit per phase
- Cost per phase
  - Build residual graph: O(m)
  - Find s-t path in residual: O(m)

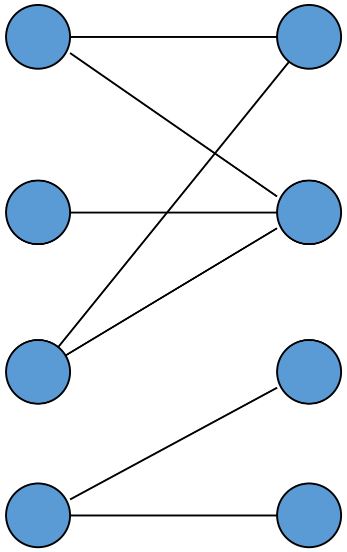


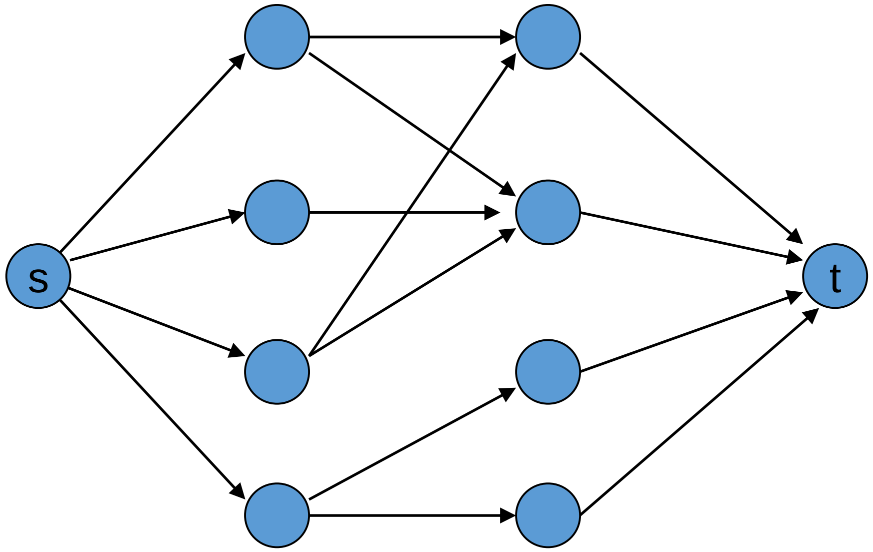
Better methods of finding augmenting paths

- Find the maximum capacity augmenting path
  - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn \log n)$ time algorithm for network flow

Network Flow Applications

Converting Matching to Network Flow





Resource Allocation: Assignment of reviewers

- A set of papers $P_1, . . ., P_n$
- A set of reviewers $R_1, . . ., R_m$
- Paper P_i requires A_i reviewers
- Reviewer R_j can review B_j papers
- For each reviewer R_j, there is a list of paper $L_{j1}, . . ., L_{jk}$ that R_j is qualified to review

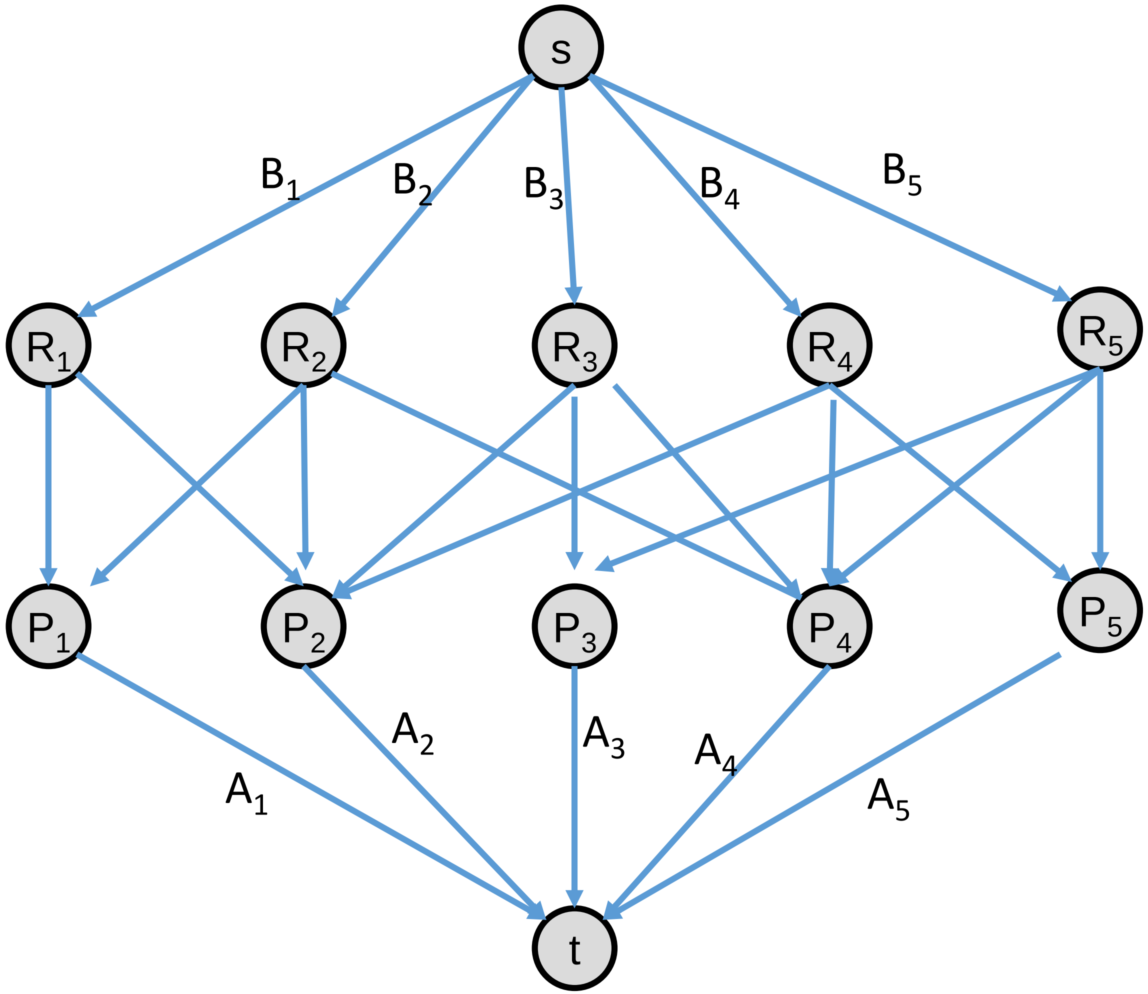
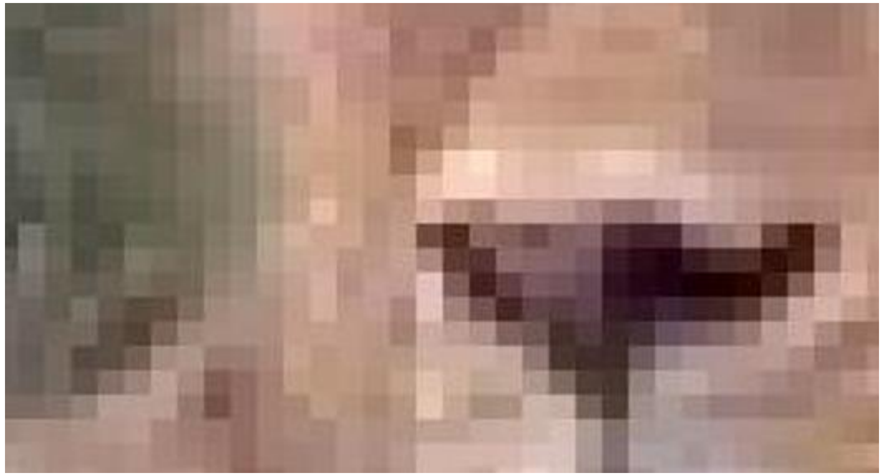
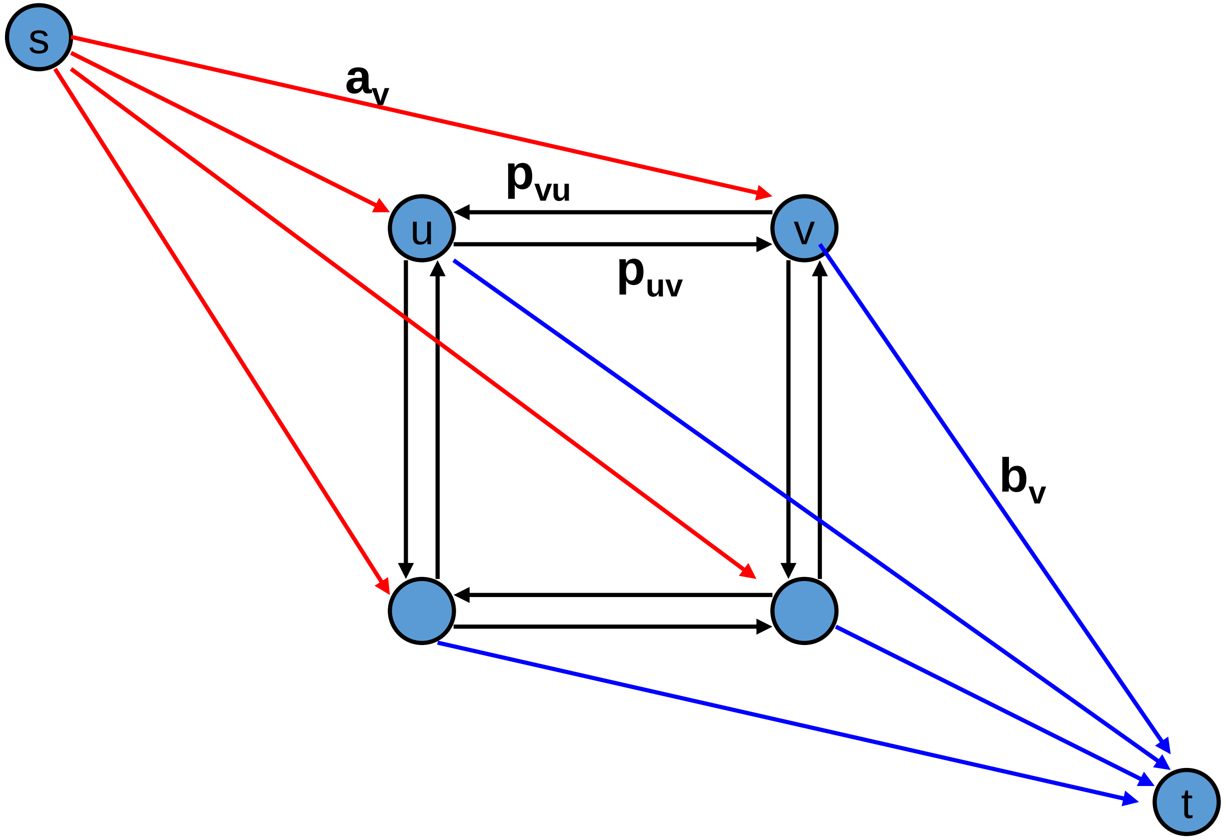


Image Segmentation

- Separate foreground from background





MinCost Network Flow Problem

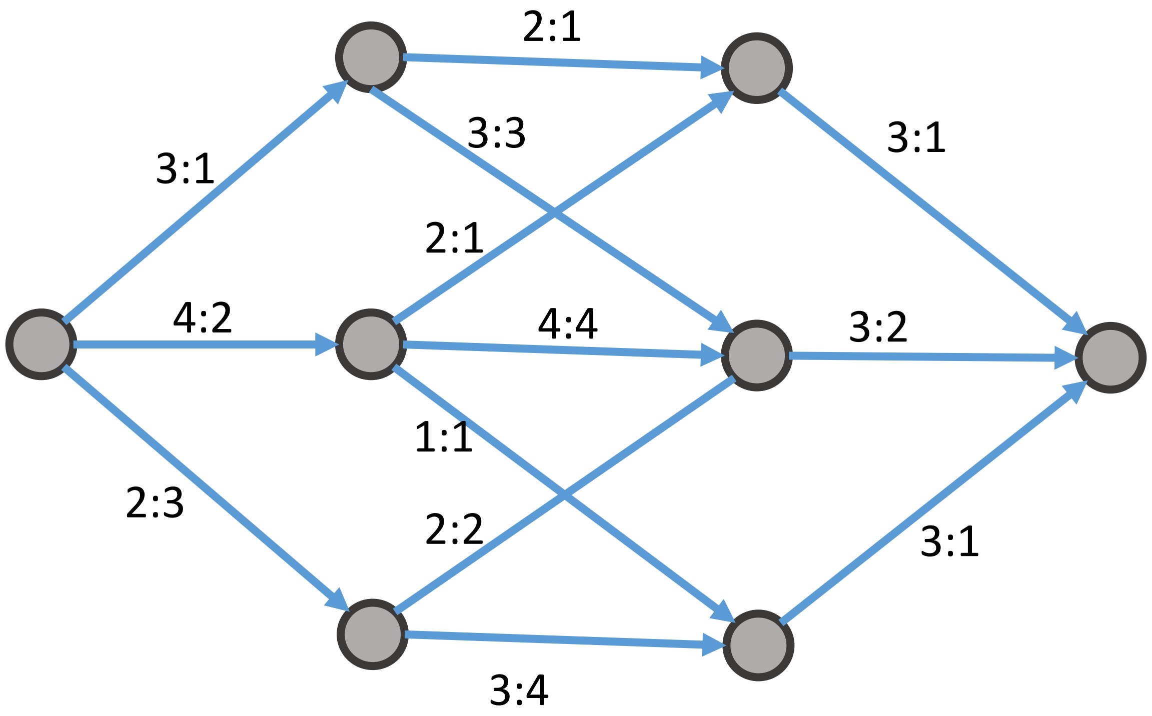
- Directed graph with source and sink
- Each edge has a capacity c(e) and a cost d(e)
- A total flow value K is specified
- Find a flow function f(e) on the edges such that
  - $0 \leq f(e) \leq c(e)$
  - Sum of the flows leaving the source is K
  - Flow is conserved at the vertices
  - The cost of the flow, $\sum_e f(e)d(e)$, is minimized

Applications of min cost flow

- Transportation problems – taking cost into account
  - An oil company is charged for using pipeline
- Allocation problems
  - Account for costs and profits
    - Showing internet ads
      - Certain number of ads of different types are available to show
      - Ads are required to reach certain demographics
      - Different profits associated with different users

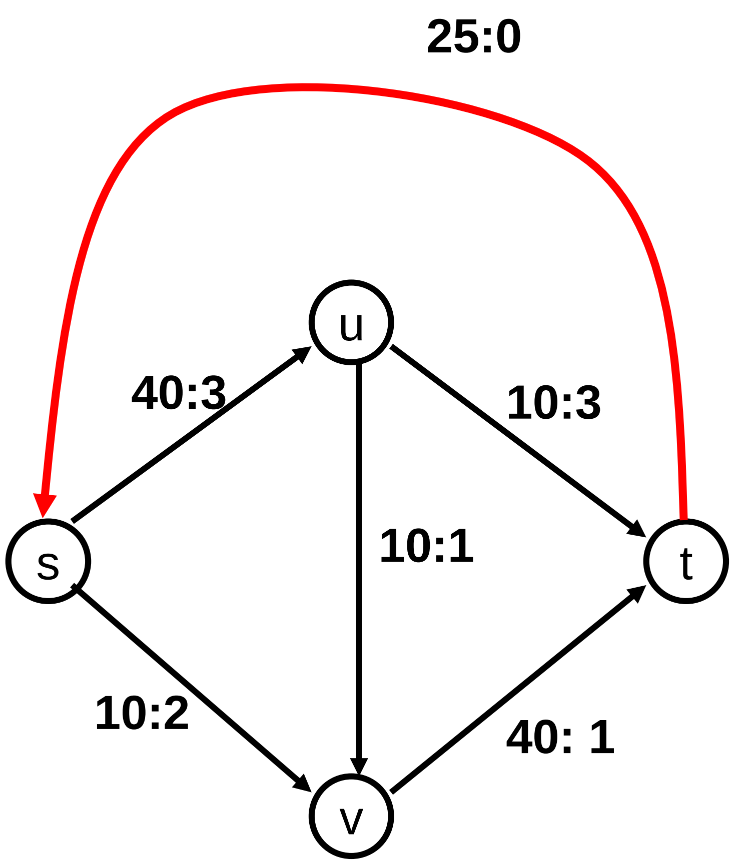
Mincost Flow

$K = 7$



Solving MinCost Flow

- Circulation – a flow problem with no source or sink
  - Add an edge from t to s with capacity K
- Find a flow of size K (K units leaving s)
- Build residual graph G_R (except for (t,s))
  - G: edge e from u to v with capacity c, cost d, and flow f
  - G_R: edge e' from u to v with capacity c – f, cost d
  - G_R: edge e'' from v to u with capacity f, cost -d



Mincost flow algorithm

```
while not done
        Construct residual graph G_R
        Find negative cost cycle C in G_R with capacity b > 0
        Add b units along C in G
```

- Adding flow along a cycle preserves the conservation of flow
- With negative cost cycles the cost keeps decreasing
- We could prove that when this stops, it has an optimal solution
- Basic algorithm needs serious engineering to make it practical