Reduction of SUBSET-SUM-OPTIMIZATION to SUBSET-SUM-DECISION

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Subset Sum

- The Subset Sum problem involves searching through a collection of numbers to find a subset that sums to a certain number.
- The Subset Sum problem is known to be NP-complete.

SUBSET-SUM-DECISION

- Problem statement:
  - Input:
    - A collection of nonnegative integers *A*
    - A nonnegative integer *b*
  - Output:
    - Boolean value indicating whether some subset of the collection sums to *b*

SUBSET-SUM-DECISION Example

- Suppose you are given as inputs:
  - The collection $A = \{2, 3, 5, 7, 10\}$
  - The sum 14
- The output is:
  - TRUE
  - $14 = 2 + 5 + 7$

SUBSET-SUM-OPTIMIZATION

- Problem statement:
  - Input:
    - A collection of nonnegative integers A
    - A nonnegative integer b
  - Output:
    - A nonnegative integer b', which is the largest integer such that:
      - $b' \leq b$, and
      - Subset-Sum-Decision(A, b') is TRUE

SUBSET-SUM-OPTIMIZATION Example

- Suppose you are given as inputs:
  - The collection $A = \{2, 3, 5, 7, 10\}$
  - The sum 16
- The output is:
  - 15
  - $15 = 5 + 10$

Reduction Requirements

- The purpose of the reduction is to write an algorithm for SUBSET-SUM-OPTIMIZATION which uses SUBSET-SUM-DECISION as an oracle.
- A good reduction should run in polynomial time using the oracle.

Naïve Approach #1

- A brute force search though all combinations of the collection *A* will take exponential time.
- Any solution that involves guessing elements to remove from *A* will probably take exponential time.
- This approach doesn't take advantage of the power of the oracle.

Naïve Approach #2

- Enumeration of the domain of b also takes exponential time.
  - The number b can be expressed with $O(\log b)$ bits.
  - There are $(b + 1)$ integers to visit.
  - $(b + 1)$ is exponential with respect to $\log b$.

Reduction Solution

- The algorithm is only a few lines long.

SUBSET-SUM-OPTIMIZATION(A, b)

    for $i \leftarrow$ floor($\log_2 b$) **downto** 0 **do**

        $A \leftarrow A + \{ 2^i \}$

    for $i \leftarrow$ floor($\log_2 b$) **downto** 0 **do**

        $A \leftarrow A - \{ 2^i \}$

        if not SUBSET-SUM-DECISION(A, b) **then**

            $b \leftarrow b - 2^i$

    return b

Adding Powers of Two

- The first step is to enumerate all the powers of two up to *b* and add them to *A*.

```
SUBSET-SUM-OPTIMIZATION(A, b)
    for i ← floor(log2 b) downto 0 do
        A ← A + { 2^i }
    for i ← floor(log2 b) downto 0 do
        A ← A - { 2^i }
        if not SUBSET-SUM-DECISION(A, b) then
            b ← b - 2^i
    return b
```

Reduction Main Loop

- The next step is the main loop.
- Each power of two is removed from *A*.

SUBSET-SUM-OPTIMIZATION(A, b)

for $i \leftarrow$ floor($\log_2 b$) **downto** 0 **do**

$A \leftarrow A + \{ 2^i \}$

for $i \leftarrow$ floor($\log_2 b$) **downto** 0 **do**

$A \leftarrow A - \{ 2^i \}$

if not SUBSET-SUM-DECISION(A, b) **then**

$b \leftarrow b - 2^i$

return b

Loop Invariants

- There are two loop invariants that allow the algorithm to work.

1. b is always greater than or equal to the optimal solution.
2. A contains a subset sum to b.

No Sum Exists Condition

- When the oracle returns FALSE, the largest valid solution is $(b - 2^i)$.

```
SUBSET-SUM-OPTIMIZATION(A, b)
    for i ← floor(log₂b) downto 0 do
        A ← A + { 2^i }
    for i ← floor(log₂b) downto 0 do
        A ← A - { 2^i }
        if not SUBSET-SUM-DECISION(A, b) then
            b ← b - 2^i
    return b
```

Return the Optimal Sum

- Finally the original collection *A* is restored.
- By this time *b* is optimal.

```
SUBSET-SUM-OPTIMIZATION(A, b)
    for i ← floor(log_2 b) downto 0 do
        A ← A + { 2^i }
    for i ← floor(log_2 b) downto 0 do
        A ← A - { 2^i }
        if not SUBSET-SUM-DECISION(A, b) then
            b ← b - 2^i
    return b
```

Execution Example

Initial Values:

- $A' = \{1, 5, 21\}$
- $b = 15$

Execution Example

Adding powers of two:

- $A' = \{1, 5, 21, 8, 4, 2, 1\}$
- $b = 15$

Execution Example

Main loop initialization:

- A' = {1, 5, 21, 8, 4, 2, 1}
- $b = 15$
- $i = 3$, $2^i = 8$

Execution Example

Remove the power of two:

- A' = {1, 5, 21, $\cancel{8}$, 4, 2, 1}
- $b = 15$
- $i = 3$, $2^i = 8$

Execution Example

Try the oracle:

- A' = {1, 5, 21, 4, 2, 1}
- $b = 15$
- $i = 3$, $2^i = 8$
- SUBSET-SUM-DECISION(A', b) = FALSE

Execution Example

No sum exists:

- $A' = \{1, 5, 21, 4, 2, 1\}$
- $b = 15 - 8 = 7$
- $i = 3,\ 2^i = 8$

Execution Example

Next loop iteration:

- A' = {1, 5, 21, 4, 2, 1}
- $b = 7$
- $i = 2$, $2^i = 4$

Execution Example

Remove the power of two:

- $A' = \{1, 5, 21, \cancel{4}, 2, 1\}$
- $b = 7$
- $i = 2$, $2^i = 4$

Execution Example

Try the oracle:

- $A' = \{1, 5, 21, 2, 1\}$
- $b = 7$
- $i = 2$, $2^i = 4$
- SUBSET-SUM-DECISION(A', b) = TRUE

Execution Example

Next loop iteration:

- A' = {1, 5, 21, 2, 1}
- $b = 7$
- $i = 1$, $2^i = 2$

Execution Example

Remove the power of two:

- $A' = \{1, 5, 21, \cancel{2}, 1\}$
- $b = 7$
- $i = 1$, $2^i = 2$

Execution Example

Try the oracle:

- A' = {1, 5, 21, 1}
- $b = 7$
- $i = 1$, $2^i = 2$
- SUBSET-SUM-DECISION(A', b) = TRUE

Execution Example

Next loop iteration:

- $A' = \{1, 5, 21, 1\}$
- $b = 7$
- $i = 0, 2^i = 1$

Execution Example

Remove the power of two:

- $A' = \{1, 5, 21, \cancel{1}\}$
- $b = 7$
- $i = 0$, $2^i = 1$

Execution Example

Try the oracle:

- A' = {1, 5, 21}
- $b = 7$
- $i = 0$, $2^i = 1$
- SUBSET-SUM-DECISION(A', b) = FALSE

Execution Example

No sum exists:

- $A' = \{1, 5, 21\}$
- $b = 7 - 1 = 6$
- $i = 0$, $2^i = 1$

Execution Example

Return the optimal sum:

- $A' = \{1, 5, 21\}$
- $b = 6$

Summary

- Each loop has O(log *b*) iterations, which is linear with respect to the size of *b*.
- The correct solution takes advantage of the NP-complete power of the oracle.