CSEP 521
Applied Algorithms
Spring 2005

Linear Programming

Reading

- Chapter 29

Outline for Tonight

- Examples of Linear Programming
- Reductions to Linear Programming
- Duality Theorem
- Approximation algorithms using LP
- Simplex Algorithm

Linear Programming

- The process of minimizing a linear objective function subject to a finite number of linear equality and inequality constraints.
- The word “programming” is historical and predates computer programming.
- Example applications:
  - airline crew scheduling
  - manufacturing and production planning
  - telecommunications network design
- “Few problems studied in computer science have greater application in the real world.”

An Example: The Diet Problem

- A student is trying to decide on lowest cost diet that provides sufficient amount of protein, with two choices:
  - steak: 2 units of protein/pound, \$3/pound
  - peanut butter: 1 unit of protein/pound, \$2/pound
- In proper diet, need 4 units protein/day.

Let x = # pounds peanut butter/day in the diet.

Let y = # pounds steak/day in the diet.

Goal: minimize $2x + 3y$ (total cost)

subject to constraints:

$x + 2y \geq 4$

$x \geq 0, \ \ y \geq 0$

This is an LP- formulation of our problem

An Example: The Diet Problem

Goal: minimize $2x + 3y$ (total cost)

subject to constraints:

$x + 2y \geq 4$

$x \geq 0, \; y \geq 0$

- This is an optimization problem.
- Any solution meeting the nutritional demands is called a *feasible solution*
- A feasible solution of minimum cost is called the *optimal solution*.

Linear Program - Definition

A linear program is a problem with n variables $x_1,\dots,x_n$, that has:

1. A linear objective function, which must be minimized/maximized. Looks like:

   $$\max\ (\min)\ c_1x_1+c_2x_2+\dots+c_nx_n$$

2. A set of m linear constraints. A constraint looks like:

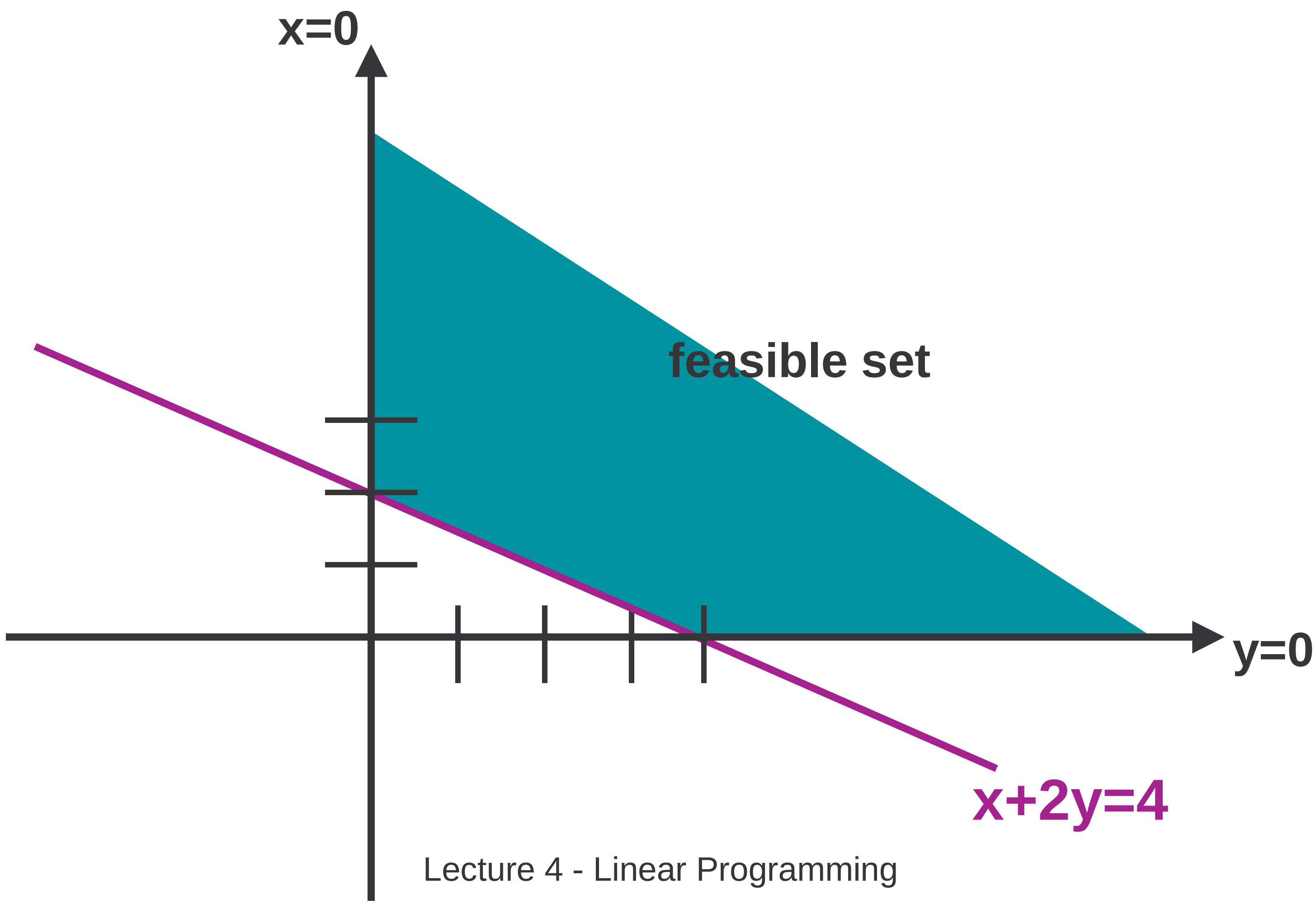
   $$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i \text{ (or } \ge \text{ or } = \text{)}$$

Note: the values of the coefficients c_i, $a_{i,j}$ are given in the problem input.

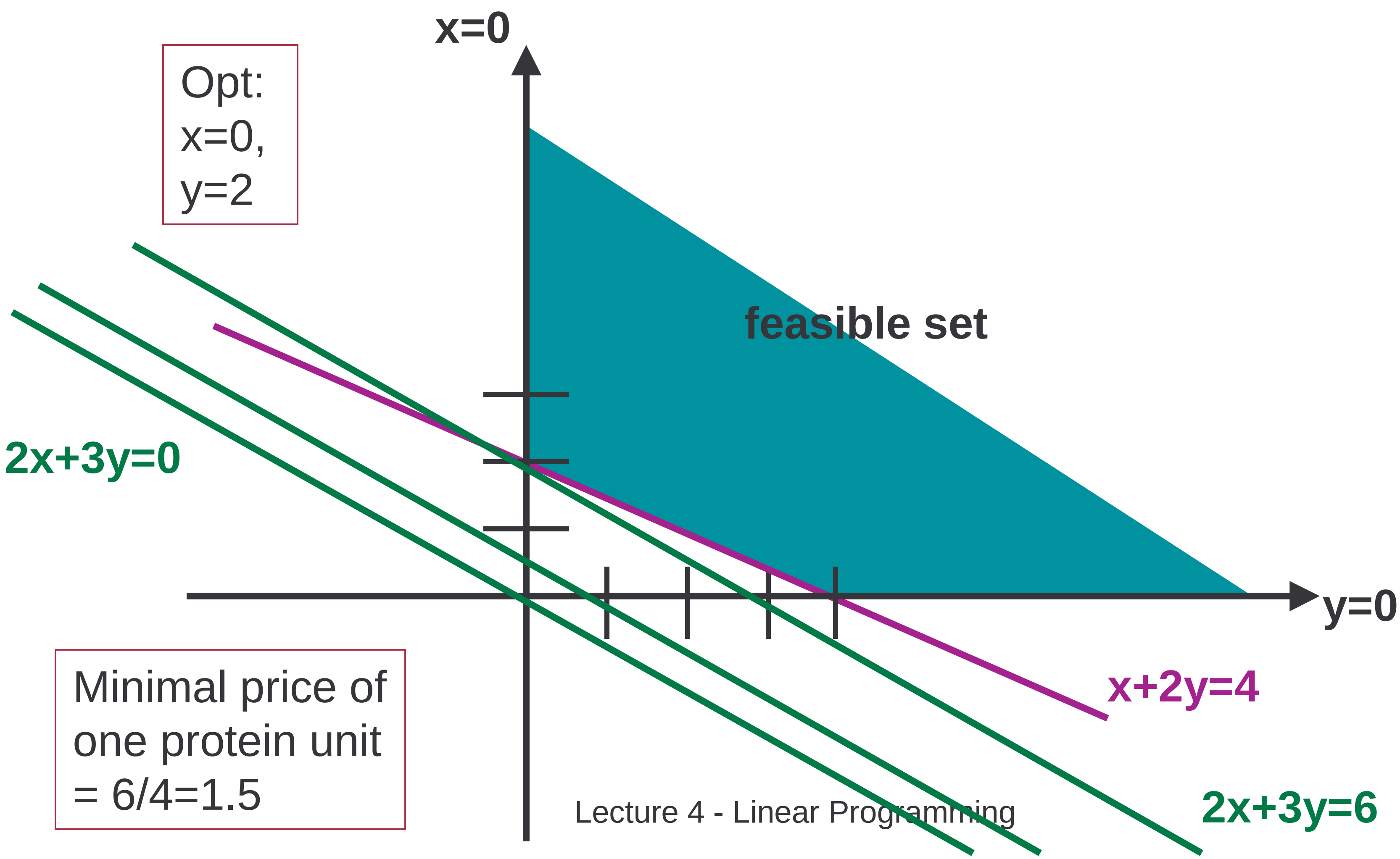
Feasible Set

- Each linear inequality divides n-dimensional space into two halfspaces, one where the inequality is satisfied, and one where it's not.
- Feasible Set : solutions to a family of linear inequalities.
- The linear cost functions, defines a family of parallel hyperplanes (lines in 2D, planes in 3D, etc.). Want to find one of minimum cost à must occur at corner of feasible set.

Visually…
x= peanut butter, y = steak

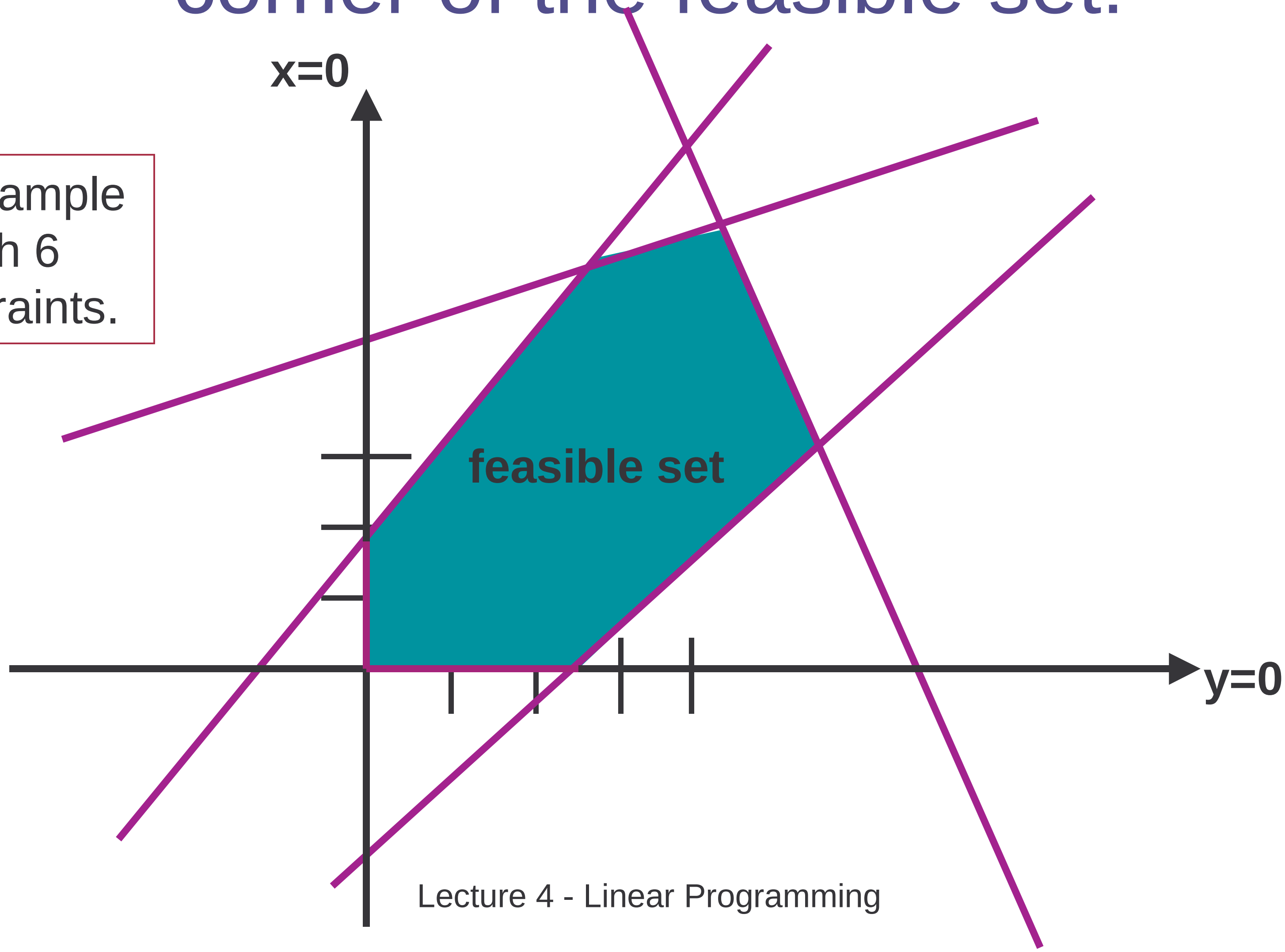


Optimal vector occurs at some corner of the feasible set!



Optimal vector occurs at some corner of the feasible set!

An Example with 6 constraints.



Standard Form of a Linear Program.

Maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to $\Sigma_{1 \leq j \leq n} a_{ij}x_j \leq b_i$ $\quad$ i=1..m

$x_j \geq 0$ $\quad$ j=1..n

Maximize cx

subject to $Ax \leq b$ and $x \geq 0$

Putting LPs Into Standard Form

- Min to Max
  - Change $\Sigma c_j x_j$ to $\Sigma(-c_j)x_j$
- Change = constraints to $\leq$ and $\geq$ constraints
- Add non-negativity constraints by substituting $x'_i - x''_i$ for x_i and adding constraints $x'_i \geq 0$, $x''_i \geq 0$.
- Change $\geq$ constraints to $\leq$ constraints by using negation

The Feasible Set of Standard LP

- Intersection of a set of half-spaces, called a polyhedron.
- If it’s bounded and nonempty, it’s a polytope.

There are 3 cases:

- feasible set is empty.
- cost function is unbounded on feasible set.
- cost has a maximum on feasible set.

First two cases very uncommon for real problems in economics and engineering.

Solving LP

- There are several polynomial-time algorithms that solve any linear program optimally.
  - Ø The Simplex method (later) (not polynomial time)
  - Ø The Ellipsoid method (polynomial time)
  - Ø More
- These algorithms can be implemented in various ways.
- There are many existing software packages for LP.
- It is convenient to use LP as a “black box” for solving various optimization problems.

LP formulation: another example

Bob's bakery sells bagel and muffins.

To bake a dozen bagels Bob needs 5 cups of flour, 2 eggs, and one cup of sugar.

To bake a dozen muffins Bob needs 4 cups of flour, 4 eggs and two cups of sugar.

Bob can sell bagels in $10/dozen and muffins in $12/dozen.

Bob has 50 cups of flour, 30 eggs and 20 cups of sugar.

How many bagels and muffins should Bob bake in order to maximize his revenue?

LP formulation: Bob's bakery

	Bagels	Muffins	Avail.
Flour	5	4	50
Eggs	2	4	30
Sugar	1	2	20
Revenue	10	12	

$$A = \begin{pmatrix} 5 & 4 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$$

$$b = \begin{bmatrix} 10 & 12 \end{bmatrix} \qquad c = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

Maximize $10x_1+12x_2$

s.t. $5x_1+4x_2 \leq 50$

$2x_1+4x_2 \leq 30$

$x_1+2x_2 \leq 20$

$x_1 \geq 0,\ x_2 \geq 0$

Maximize $b \cdot x$

s.t. $Ax \leq c$

$x \geq 0.$

In class exercise: Formulate as LP

You want to invest $1000 in 3 stocks, at most $400 per stock

	price/share	dividends/year
stock A	$50	$2
stock B	$200	$5
stock C	$20	0

Stock C has probability ½ of appreciating to $25 in a year, and prob ½ of staying $20.

What amount of each stock should be bought to maximize dividends + expected appreciation over a year?

In class exercise: Formulate as LP

Solution: Let x_a, x_b, and x_c denote the amounts of A,B,C stocks to be bought.

Objective function:

Constraints:

Reduction Example: Max Flow

Max Flow is reducible to LP

Variables: $f(e)$ - the flow on edge e.

Max $\Sigma_{e \in out(s)} f(e)$ (assume s has zero in-degree)

Subject to

$f(e) \leq c(e), \ \forall e \in E$ (Edge condition)

$\Sigma_{e \in in(v)} f(e) - \Sigma_{e \in out(v)} f(e) = 0 \ , \forall v \in V-\{s,t\}$

(Vertex condition)

$f(e) \geq 0, \ \forall e \in E$

Example - L_1 Best Fit

$$\sum_{i=1}^{n} a_{ij} x_i = b_j \quad 1 \leq j \leq m$$

Example:
2x + 3y = 6
3x - 5y = 2
4x + 5y = 7

- If m > n then overconstrained.
- Find $x_1,\ldots,x_n$ to minimize

$$\sum_{j=1}^{m} \left| \sum_{i=1}^{n} a_{ij} x_i - b_j \right|$$

L_1 norm

L_1 Best Fit Reduced to LP

$$\text{minimize } \sum_{j=1}^{m} e_j$$

$$\text{subject to } b_j - \sum_{i=1}^{n} a_{ij} x_i \le e_j$$

$$\sum_{i=1}^{n} a_{ij} x_i - b_j \le e_j$$

$$e_j \ge 0$$

$$\text{for } 1 \le j \le m$$

A Central Result of LP Theory: Duality Theorem

- Every linear program has a dual
- If the original is a maximization, the dual is a minimization and vice versa
- Solution of one leads to solution of other

Primal: Maximize xc subject to $Ax \leq \mathbf{b}$, $x \geq 0$

Dual: Minimize yb subject to $yA^T \geq c$, $y \geq 0$

If one has optimal solution so does other, and their values are the same.

Primal: Maximize **xc** subject to A**x** $\leq$ **b**, **x** ≥ 0

Dual: Minimize y**b** subject to $yA^T \geq$ **c**, **y** ≥ 0

- In the primal, **c** is cost function and **b** was in the constraint. In the dual, reversed.
- Inequality sign is changed and maximization turns to minimization.

Primal:

maximize $2x + 3y$

s.t $x+2y \leq 4$,

$2x + 5y \leq 1$,

$x - 3y \leq 2$,

$x \geq 0,\ y \geq 0$

Dual:

minimize $4p + q + 2r$

s.t $p+2q + r \geq 2$,

$2p+5q - 3r \geq 3$,

$p,q,r \geq 0$

Weak Duality Theorem

- Theorem: If x is a feasible solution to the primal problem and y is a feasible solution to the dual problem then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

Proof of Weak Duality

$$\begin{aligned}
\sum_{j=1}^{n} c_j x_j &\le \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j \\
&= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \\
&\le \sum_{i=1}^{m} b_i y_i
\end{aligned}$$

Duality Theorem

- If x* is optimal for the primal and y* is optimal for the dual, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Simple Example of Duality

- Diet problem: minimize $2x + 3y$

  subject to $x+2y \geq 4$,

  $x \geq 0$, $y \geq 0$
- Dual problem: maximize $4p$

  subject to $p \leq 2$,

  $2p \leq 3$,

  $p \geq 0$
- Dual: the problem faced by a druggist who sells synthetic protein, trying to compete with peanut butter and steak

Simple Example

- The druggist wants to maximize the price p, subject to constraints:
  - synthetic protein must not cost more than protein available in foods.
  - price must be non-negative or he won’t sell any
  - revenue to druggist will be 4p
- Solution: $p \leq 3/2$ à objective value = 4p = 6
- Not coincidence that it’s equal the minimal cost in original problem.

What's going on?

- Notice: feasible sets completely different for primal and dual, but nonetheless an important relation between them.
- Duality theorem says that in the competition between the grocer and the druggist the result is always a tie.
- Optimal solution to primal tells purchaser what to do.
- Optimal solution to dual fixes the natural prices at which economy should run.
- The diet x and vitamin prices y are optimal when
  - grocer sells zero of any food that is priced above its vitamin equivalent.
  - druggist charges 0 for any vitamin that is oversupplied in the diet.

Duality Theorem

Druggist's max revenue = Purchasers min cost

Practical Use of Duality:

- Sometimes simplex algorithm (or other algorithms) will run faster on the dual than on the primal.
- Can be used to bound how far you are from optimal solution.
- Important implications for economists.

Linear Programming, Mid-Summary

- Of great practical importance to solve linear programs:
  - they model important practical problems
    - production, approximating the solution of inconsistent equations, manufacturing, network design, flow control, resource allocation.
  - solving an LP is often an important component of solving or approximating the solution to an **integer linear programming problem.**
- Can be solved in poly-time, the simplex algorithm works very well in practice.
- One problem where you really do not want to write your own code.

LP-based approximations

- We don't know any polynomial-time algorithm for any NP-complete problem
- We know how to solve LP in polynomial time
- We will see that LP can be used to get approximate solutions to some NP-complete problems.

Weighted Vertex Cover

Input: Graph $G=(V,E)$ with non-negative weights $w(v)$ on the vertices.

Goal: Find a minimum-cost set of vertices S, such that all the edges are covered. An edge is covered iff at least one of its endpoints is in S.

Recall: Vertex Cover is NP-complete.

The best known approximation factor is

$2-(\log \log |V|/2 \log|V|)$.

Weighted Vertex Cover

Variables: for each $v \in V$, $x(v)$ – is v in the cover?

Min $\Sigma_{v \in V}\, w(v)x(v)$

s.t.

$x(v) + x(u) \geq 1, \quad \forall (u,v) \in E$

$x(v) \in \{0,1\} \quad \forall v \in V$

The LP Relaxation

This is **not** a linear program: the constraints of type $x(v) \in \{0,1\}$ are not linear.

Such problems (LP's with integrality constraints on variables) are called **integer linear programs (IP).** Solving IP's is an NP-hard problem.

However, if we replace the constraints $x(v) \in \{0,1\}$ by $x(v) \geq 0$ and $x(v) \leq 1$, we will get a linear program.

The resulting LP is called a **Linear Relaxation** of IP, since we relax the integrality constraints.

LP Relaxation of Weighted Vertex Cover

Min $\Sigma_{v\in V}\, w(v)x(v)$

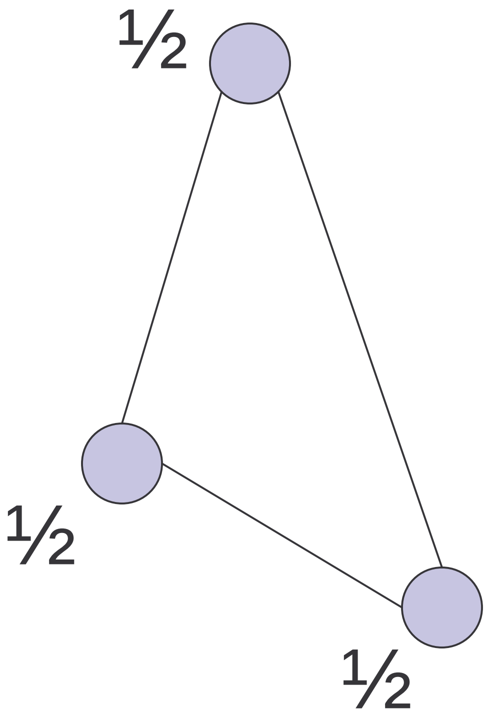
s.t.

$x(v) + x(u) \geq 1, \;\; \forall (u,v)\in E$

$x(v)\geq 0, \;\; \forall v\in V$

$x(v) \leq 1, \;\; \forall v\in V$

LP Relaxation of Weighted Vertex Cover - example



Consider the case in which all weights are 1.

An optimal VC has cost 2 (any two vertices)

An optimal relaxation has cost 3/2 (for all three vertices x(v)=1/2)

The LP and the IP are different problems. Can we still learn something about Integer VC?

Why LP Relaxation Is Useful ?

The optimal value of LP-solution provides a bound on the optimal value of the original optimization problem. OPT(LP) is always better than OPT(IP) (why?)

Therefore, if we find an integral solution within a factor r of OPT_{LP}, it is also an r-approximation of the original problem.

These can be done by "wise" rounding.

Approximation of Vertex Cover Using LP-Rounding

1. Solve the LP-Relaxation.

2. Let S be the set of all the vertices v with $x(v) \geq 1/2$. Output S as the solution.

Analysis: The solution is feasible: for each edge $e=(u,v)$, either $x(v) \geq 1/2$ or $x(u) \geq 1/2$

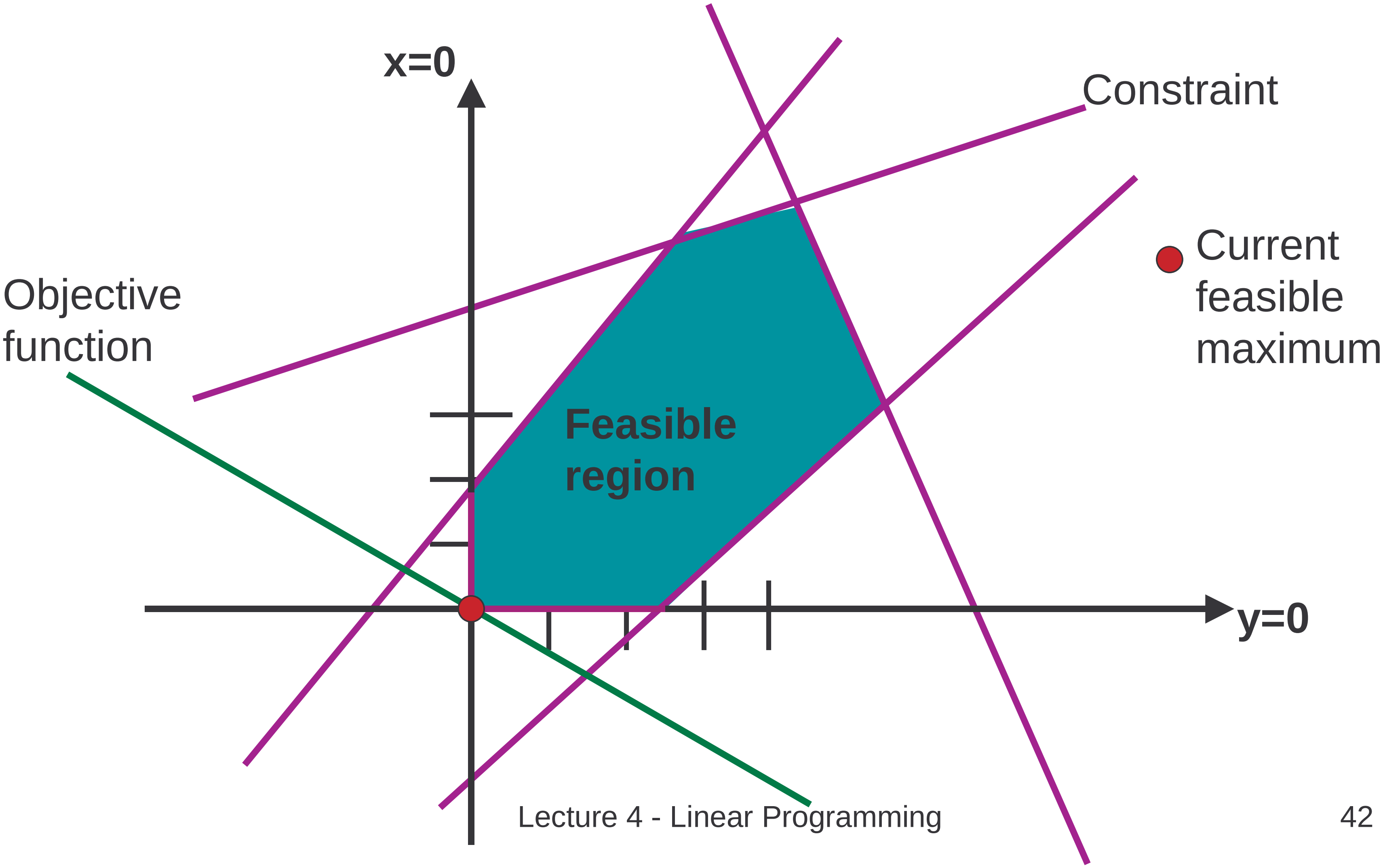
The value of the solution is: $\Sigma_{v\in S}\, w(v) = \Sigma_{\{v|x(v)\ \geq 1/2\}}\, w(v) \leq 2\Sigma_{v\in V}\, w(v)x(v) = 2OPT_{LP}$

Since $OPT_{LP} \leq OPT_{VC}$, the cost of the solution is $\leq 2OPT_{VC}$.

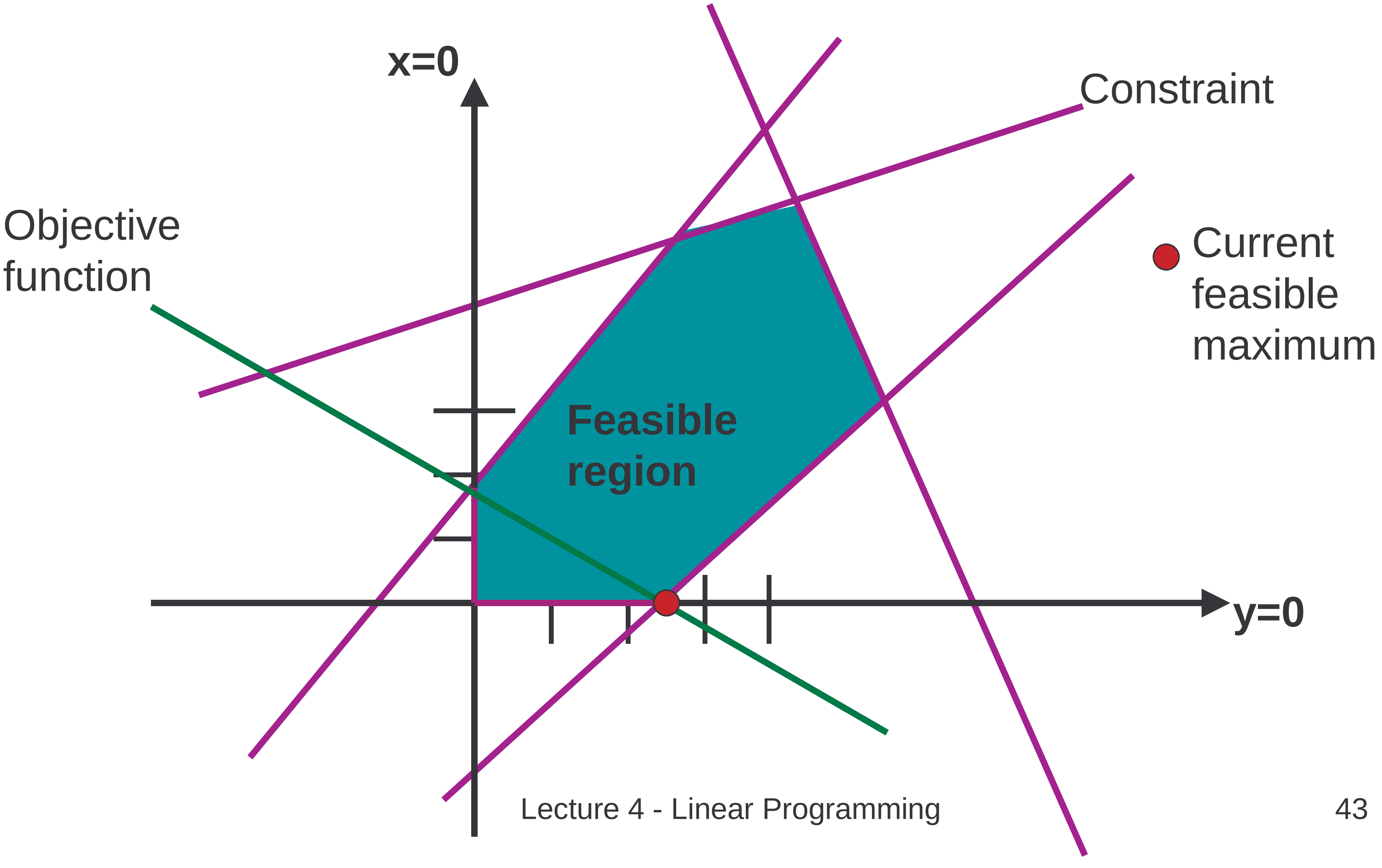
The Simplex Method

- Start with LP into standard form.
- Phase I : Assume x = 0 is a feasible solution, that is, 0 is in the feasible region. If not, then an auxiliary simplex method is used to start find a feasible solution (more later).
- Phase II: Use the “slack” version of the LP to move from corner to corner along the edges of the feasible region. Technically, we’ll see how to move from one slack version of the LP to another to achieve this.
- When reach a local maximum you’ve found the optimum.

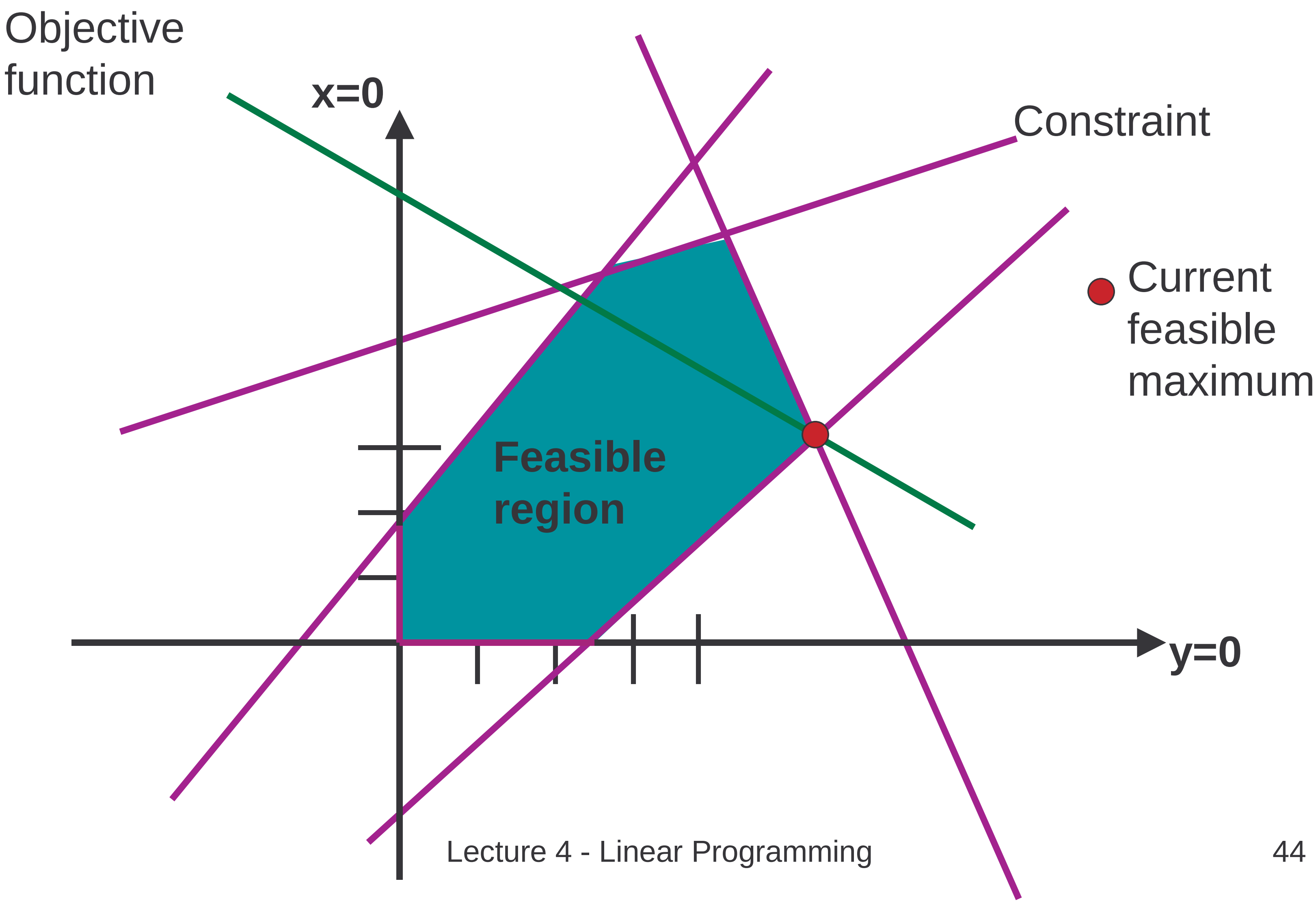
Phase II


x=0
Constraint
Current feasible maximum
Objective function
Feasible region
y=0

Phase II


x=0
Constraint
Objective
function
Current
feasible
maximum
Feasible
region
y=0

Phase II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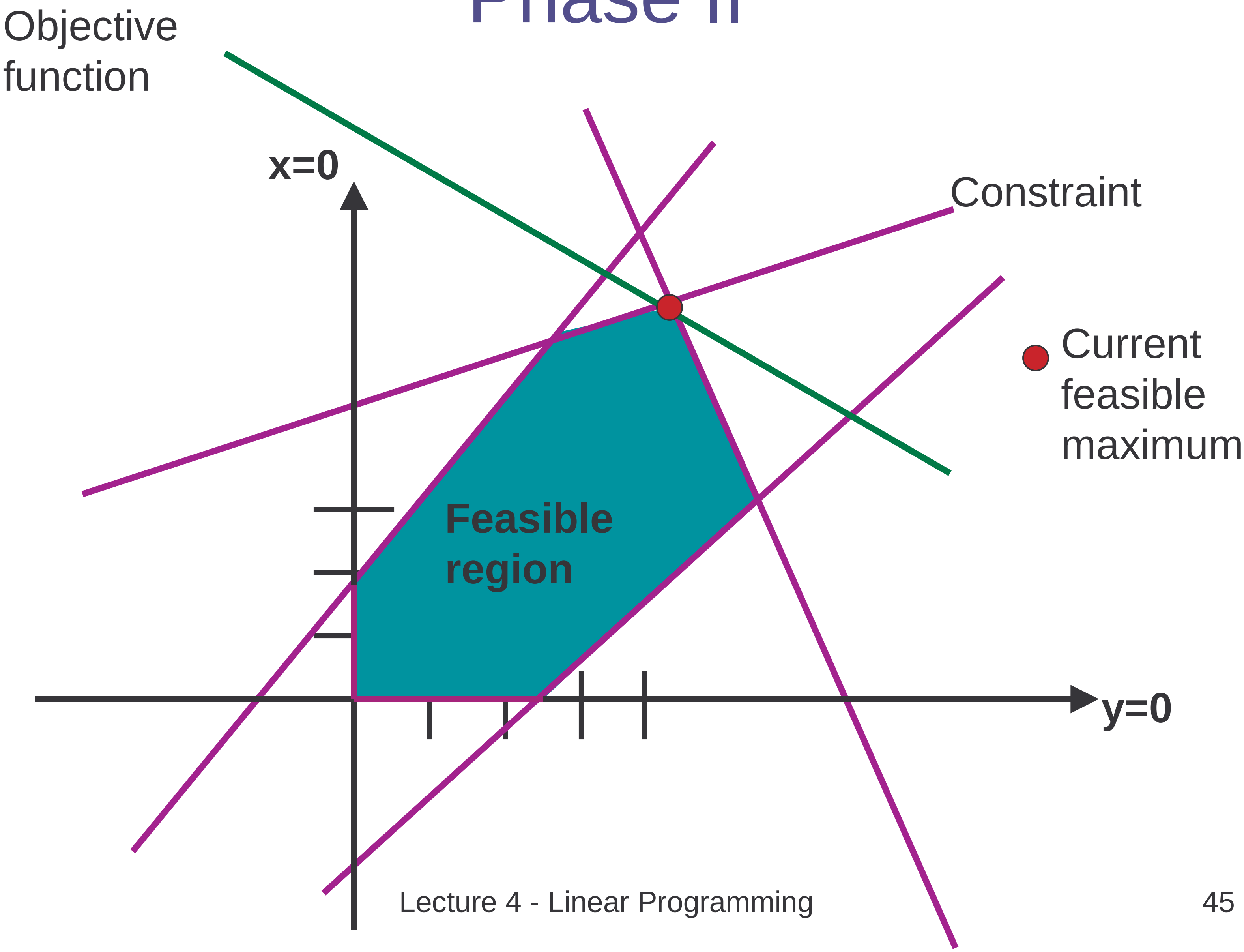
Objective
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Simplex Algorithm: An Example in 3D

Maximize $5x + 4y + 3z$

subject to $2x + 3y + z \leq 5$

$4x + y + 2z \leq 11$

$3x + 4y + 2z \leq 8$

$x,y,z \geq 0.$

Convert inequalities into equalities by introducing slack variables a,b,c.

Define: $a = 5-2x-3y-z$ à $a \geq 0$

$b = 11-4x-y-2z$ à $b \geq 0$

$c = 8-3x-4y-2z$ à $c \geq 0$

$F = 5x+4y+3z$, objective function

Initial State of Simplex

$$\overbrace{x=y=z=0}^{\text{Nonbasic}},\ \overbrace{a=5,\ b=11,\ c=8}^{\text{Basic}},\ \overbrace{F=0}^{\text{Objective}}$$

$a = 5-2x-3y-z$

$b = 11-4x-y-2z$

$c = 8-3x-4y-2z$

$\mathbf{F = 5x+4y+3z}$

Choose a Pivot and Tighten

$$\underbrace{x=y=z=0}_{\text{Nonbasic}},\ \underbrace{a=5,\ b=11,\ c=8}_{\text{Basic}},\ \underbrace{F=0}_{\text{Objective}}$$

$a = 5-2x-3y-z$ $\quad x \leq 5/2$ most stringent!

$b = 11-4x-y-2z$ $\quad x \leq 11/4$

$c = 8-3x-4y-2z$ $\quad x \leq 8/3$

$F = 5x+4y+3z$

Pivot is x

Remove Pivot From RHS

$$\overbrace{a=y=z=0}^{\text{Nonbasic}},\ \overbrace{x=5/2,\ b=1,\ c=1/2}^{\text{Basic}},\ \overbrace{F=25/2}^{\text{Objective}}$$

$a = 5-2x-3y-z$ Ł $x = 5/2 -3/2\,y -1/2\,z - 1/2\,a$

$b = 11-4x-y-2z$ Ł $b = 1 + 5y +2a$

$c = 8-3x-4y-2z$ Ł $c = 1/2 +1/2\,y -1/2\,z + 3/2\,a$

$F = 5x+4y+3z$ Ł $F= 25/2-7/2\,y +1/2\,z -5/2\,a$

Choose a Pivot and Tighten

$$\underbrace{a=y=z=0}_{\text{Nonbasic}},\ \underbrace{x=5/2,\ b=1,\ c=1/2}_{\text{Basic}},\ \underbrace{F=25/2}_{\text{Objective}}$$

$x = 5/2 \;\; -3/2\, y -1/2\, z - 1/2\, a \qquad z \leq 5$

$b = 1 \quad + \quad 5y \qquad\qquad\qquad +2a \qquad z \leq \infty$

$c = 1/2 +1/2\, y -1/2\, z + 3/2\, a \qquad z \leq 1$ most stringent!

$F= 25/2-7/2\, y +1/2\, z -5/2\, a$

Pivot is z

Remove Pivot from RHS

$$\overbrace{a=y=c = 0}^{\text{Nonbasic}},\ \overbrace{x = 2,\ b = 1,\ z = 1}^{\text{Basic}},\ \overbrace{F = 13}^{\text{Objective}}$$

x = 5/2 -3/2 y -1/2 z - 1/2 a Ł x = 2 - 2y - 2a + c

b = 1 + 5y +2a Ł b = 1 + 5y +2a

c = 1/2 +1/2 y -1/2 z + 3/2 a Ł z = 1 + y + 3a - 2c

F= 25/2-7/2 y +1/2 z -5/2 a Ł F = 13 - 3y - a - c

No way to increase F so we're done

Final Solution

Original Problem

Maximize $5x + 4y + 3z$

subject to $2x + 3y + z \leq 5$ tight

$4x + y + 2z \leq 11$ not tight

$3x + 4y + 2z \leq 8$ tight

$x, y, z \geq 0$.

Solution: $x = 2$, $y = 0$, $z = 1$, $F = 13$

Finding a Feasible Solution

- Suppose that 0 is not feasible, then Phase 1 finds a feasible solution in terms of the basic variables.

Maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to $\Sigma_{1 \le j \le n} a_{ij}x_j \le b_j$ i=1..m

$x_j \ge 0$ j=1..n

Ignore the objective function

Maximize $-x_0$

subject to $\Sigma_{1 \le j \le n} a_{ij}x_j - x_0 \le b_j$ i=1..m

$x_j \ge 0$ j=0..n

Feasible solution exist only if maximum has $x_0 = 0$

Example

Maximize $2x - y$

subject to $2x - y \le 2$

$x - 5y \le -4$

$x, y \ge 0.$

$x = y = 0$ is not a feasible solution

Auxiliary LP

Maximize $-z$

subject to $2x - y - z \leq 2$

$x - 5y - z \leq -4$

$x, y, z \geq 0.$

Move to Slack Form

Assuming $x=y=0$, we need to make sure $a,b \geq 0$

$a = 2 - 2x + y + z \quad z \geq -2$

$b = -4 - x + 5y + z \quad z \geq 4$ most stringent!

$F = -z$

We do a pivot step with z.

Remove Pivot from RHS

$$\overbrace{x=y=b=0,}^{\text{Nonbasic}}\ \overbrace{a = 6,\ z = 4,}^{\text{Basic}}\ \overbrace{F = -4}^{\text{Objective}}$$

a = 2 - 2x + y + z Ł a = 6 - x - 4y + b

b = -4 - x + 5y + z Ł z = 4 + x - 5y + b

F = -z Ł F = -4 - x + 5y - b

Choose a Pivot and Tighten

$$\overbrace{x=y=b=0}^{\text{Nonbasic}},\ \overbrace{a = 6,\ z = 4}^{\text{Basic}},\ \overbrace{F = -4}^{\text{Objective}}$$

$a = 6 - x - 4y + b \quad y \leq 6/4$

$z = 4 + x - 5y + b \quad y \leq 4/5$ most stringent!

$F = -4 - x + 5y - b$

Pivot is y

Remove Pivot from RHS

$$\overbrace{x=z=b=0,}^{\text{Nonbasic}}\ \overbrace{a = 14/5,\ y = 4/5,}^{\text{Basic}}\ \overbrace{F = 0}^{\text{Objective}}$$

$a = 6 - x - 4y + b$ Ł $a = 14/5 - 9/5\ x - 4/5\ z - 1/5\ b$

$z = 4 + x - 5y + b$ Ł $y = 4/5 + 1/5\ x - 1/5\ z + 1/5\ b$

$F = -4 - x + 5y - b$ Ł $F = 0 - z$

Auxiliary LP is solved because there is no way to increase F.
Force z = 0 permanently.
Start Simplex with x = b = 0, a = 14/5, y = 4/5, and
F equal to the original F with nonbasic variables on right hand side

Starting the Simplex

$$\overbrace{x=b=0}^{\text{Nonbasic}},\ \overbrace{a = 14/5,\ y = 4/5}^{\text{Basic}},\ \overbrace{F = -4/5}^{\text{Objective}}$$

$$a = 14/5 - 9/5\,x - 1/5\,b$$

$$y = 4/5 + 1/5\,x + 1/5\,b$$

$$F \underbrace{= 2x - y}_{\text{Original objective function}} = -4/5 + 9/5\,x - 1/5\,b$$

Class Exercise

- Complete the solving of this LP

$$\begin{aligned} \text{Maximize} \quad & 2x - y \\ \text{subject to} \quad & 2x - y \le 2 \\ & x - 5y \le -4 \\ & x, y \ge 0. \end{aligned}$$

Choosing a Pivot

- Largest increase in F
- Largest positive coefficient
- First positive coefficient

Termination

- Technically, simplex can loop.
- Only so many choices for basic variables. Lack of termination can be detected.
- Termination is not usually a problem

LP Summary

- LP is widely used in practice.
- LP has a very nice theory.
- LP relaxation and rounding used in approximation algorithms
- Simplex method is a good method but has some pathological exponential cases.