CSEP 521
Applied Algorithms
Spring 2005

Traveling Salesman Problem

NP-Completeness

Reading

- Chapter 34
- Chapter 35

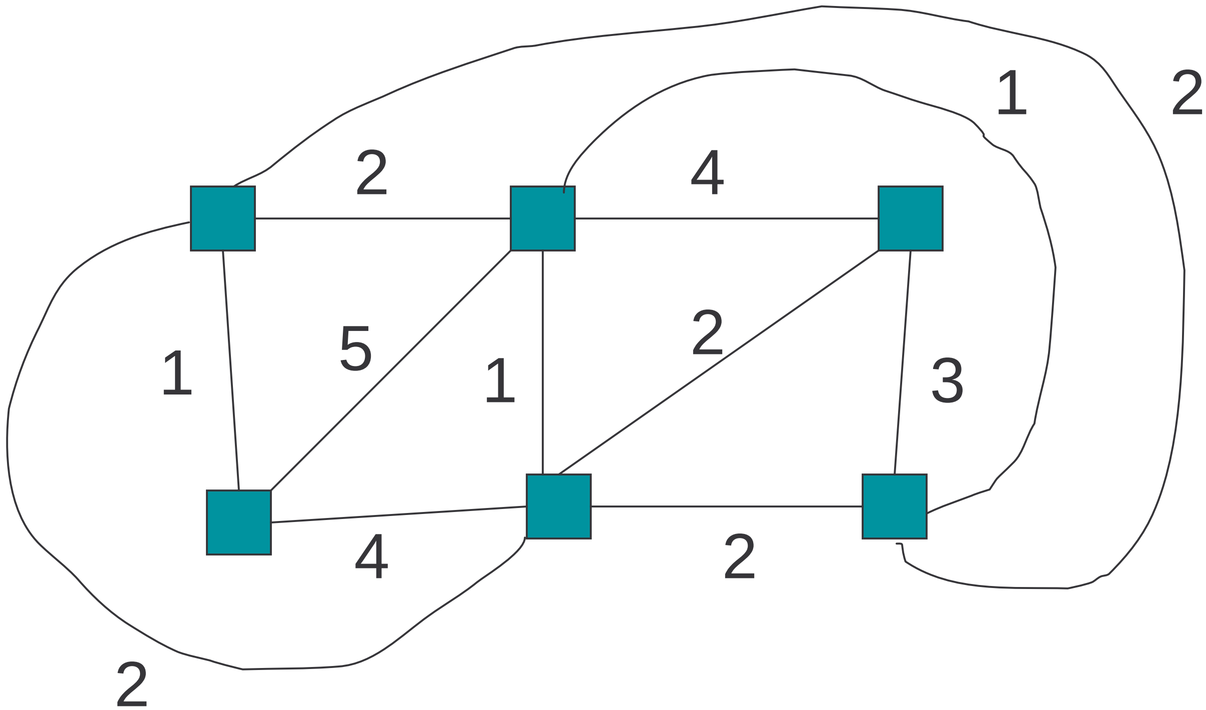
Outline for the Evening

- Traveling Salesman Problem
  - Approximation algorithms
  - Local search algorithms
- P and NP
- Reducibility and NP-Completeness
- Clique, Colorability, and other NP-complete problems
- Coping with NP-completeness

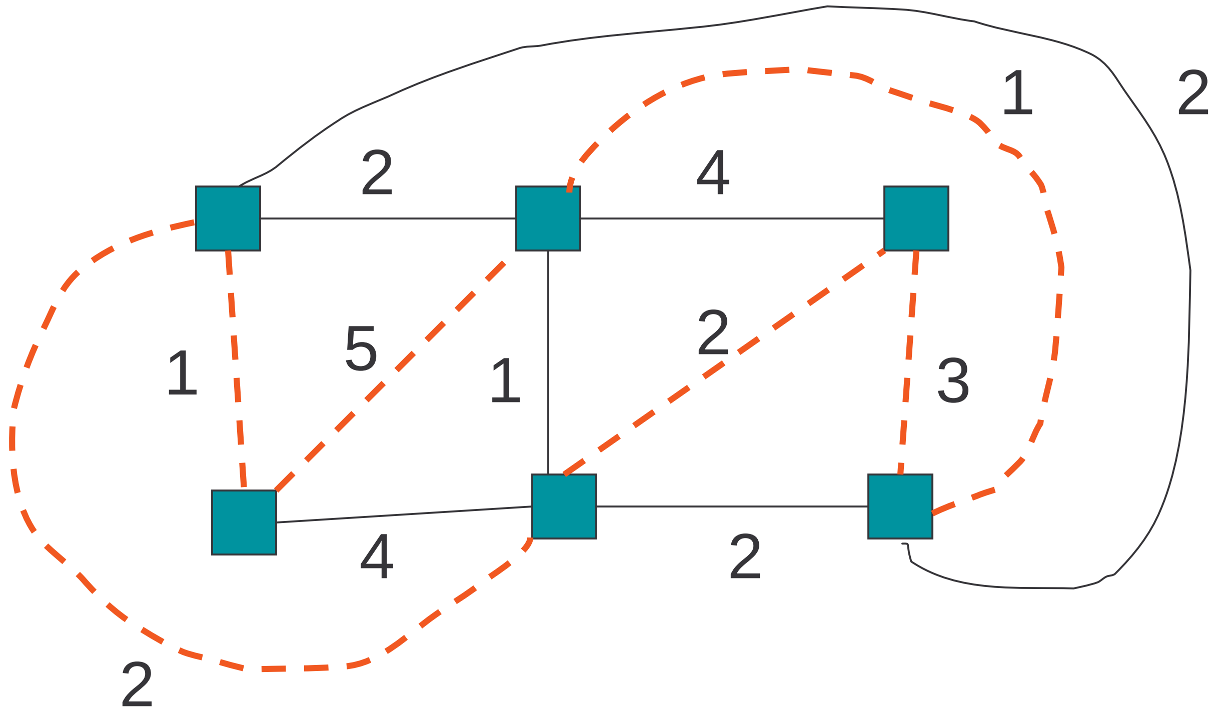
Traveling Salesman Problem

- Input: Undirected Graph G = (V,E) and a cost function C from E to the reals. C(e) is the cost of edge e.
- Output: A cycle that visits each vertex exactly once and is minimum total cost.

Example



Example



Cost = 1 + 5 + 1 + 3 + 2 + 2 = 14

Variations

- Hamiltonian Cycle
  - Is there a cycle that visits each vertex exactly once
  - Ignores costs
- Triangle inequality constraint
  - $C(u,v) \leq C(u,x) + C(x,v)$
- Euclidean Traveling Salesman
  - Vertices are points on the plane and the cost is the Euclidian distance between them
  - Implies triangle inequality

Applications

- Telescope planning
- Route planning
  - coin pickup
  - mail delivery
  - book order pickup in the Amazon warehouse
- Circuit board drilling

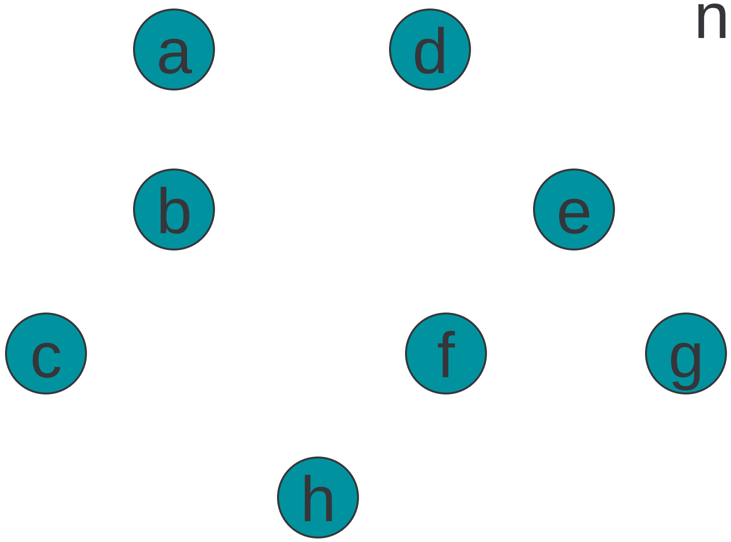
Why Traveling Salesman?

- Old well-studied problem
- Example of an NP-hard problem
  - These problems are very hard to solve exactly
  - No polynomial time algorithms known to exist
- Interesting and effective approximation algorithms
  - Good practical algorithms
  - Simple algorithms with provable approximation bounds

Approximation Alg. vs. Heuristic

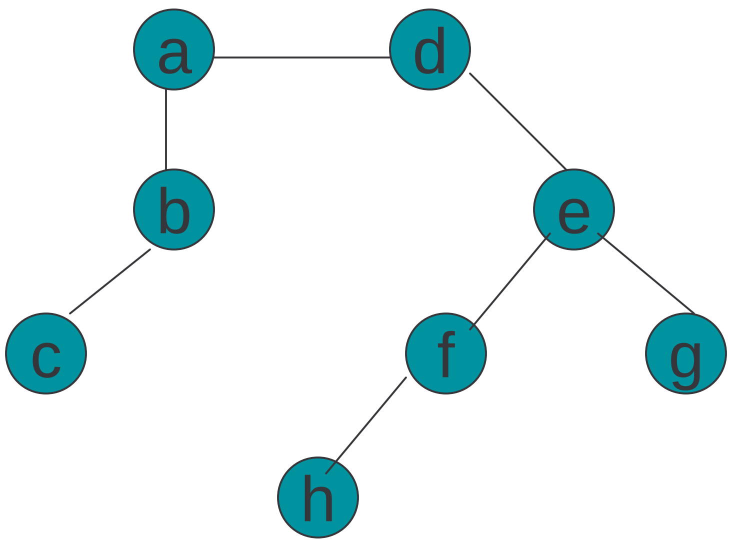
- Approximation Algorithm
  - There is a provable guarantee of how close the algorithm's result is to the optimal solution.
- Heuristic
  - The algorithm finds a solutions but there is no guarantee how good the solution is.
  - Heuristics often outperform provable approximation algorithms.

A Simple Approximation Algorithm

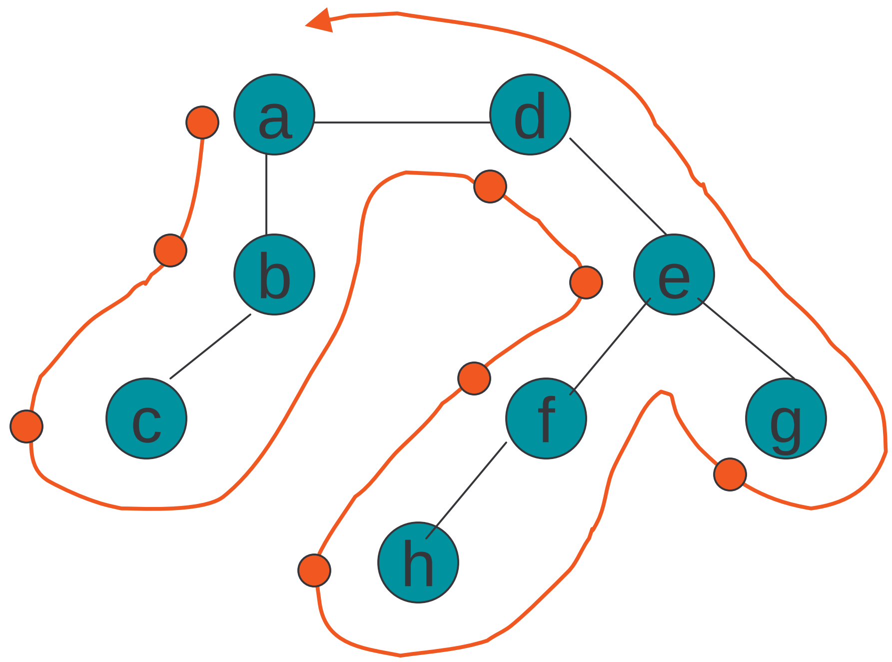


Euclidean distance
n(n-1)/2 edges

1. Find a Minimum Spanning Tree

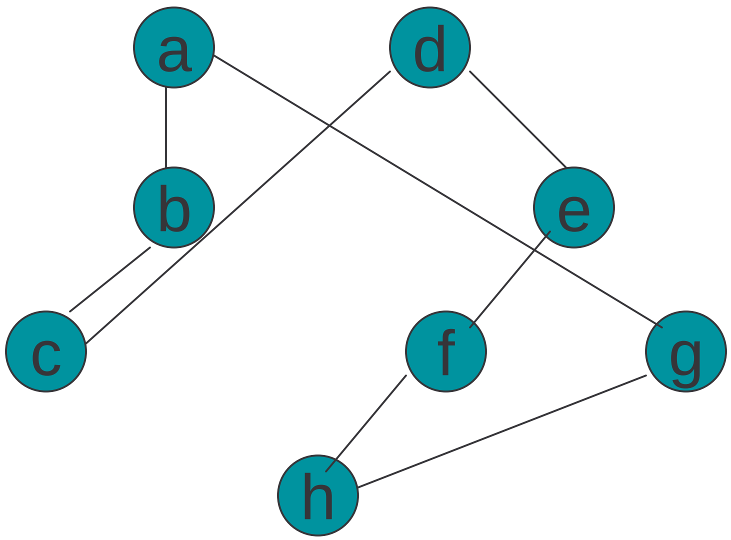


2. Depth-First Search of Tree



Marking Order = a, b, c, d, e, f, h, g

3. Connect Vertices in Marking Order



Marking Order = a, b, c, d, e, f, h, g

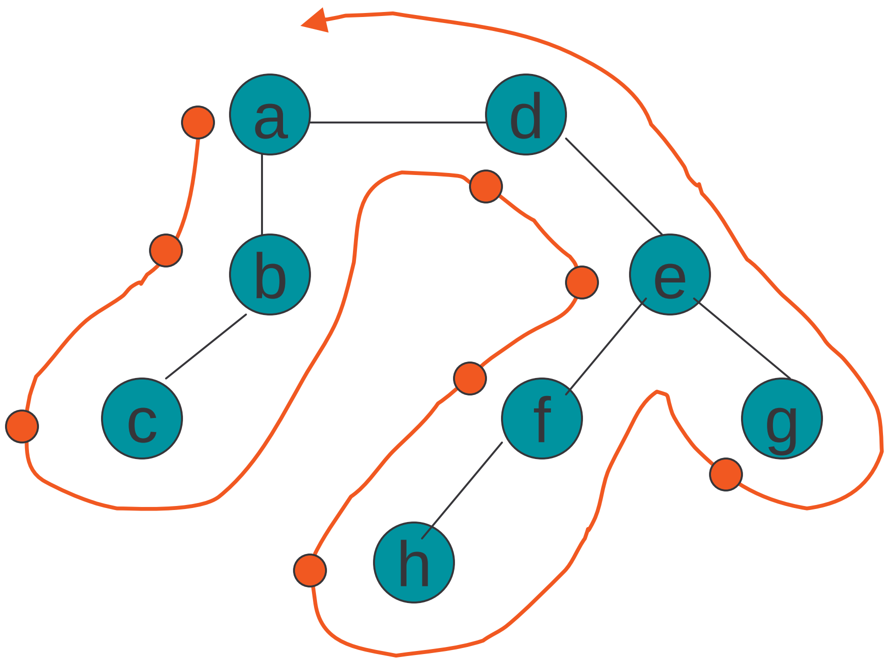
Evaluation

- Time and Storage
  - Time $O(n^2 \log n)$ with Kruskal's Algorithm
  - Storage $O(n^2)$
- Quality of Solution H
  - $C(H) \leq 2\ C(H^*)$ where H^* is an optimal tour
  - This is a "2-approximation algorithm"
- Same approximation bound applies to case of triangle inequality

Proof of Approximation Bound

- Setup
  - T minimum spanning tree
  - W the depth-first walk of T
  - H the tour computed by the algorithms
  - H* an optimal tour

Depth-First Walk



C(W) = 2 C(T)
C(H) ≤ C(W)
triangle inequality

```
Depth-first walk = a,b,c,b,a,d,e,f,h,f,e,g,e,d,a
   Marking order = a,b,c,    d,e,f,h,     g
```

Proof of Approximation Bound

1. C(W) = 2 C(T)
2. C(H) ≤ C(W), triangle inquality
3. C(H) ≤ 2 C(T), last two lines
4. C(T) ≤ C(H*), minus an edge H* is a spanning tree
5. C(H) ≤ 2 C(H*), last two lines

Solving TSP Exactly

- Branch-and-Bound
  - n < 25?
- Linear Programming
  - n < 100
- Cutting Plane Methods for Euclidian case
  - n < 15,000 with “concord”
  - see http://www.math.princeton.edu/tsp/

Solving TSP Approximately

- 3/2 – approximation algorithm of Christofedes
- Polynomial approximation scheme for Euclidian TSP by Aurora (1998), Mitchell (1999)
  - To get within $(1+\varepsilon)$ of optimal can be done in time polynomial in $1/\varepsilon$ and n.
  - These are not practical

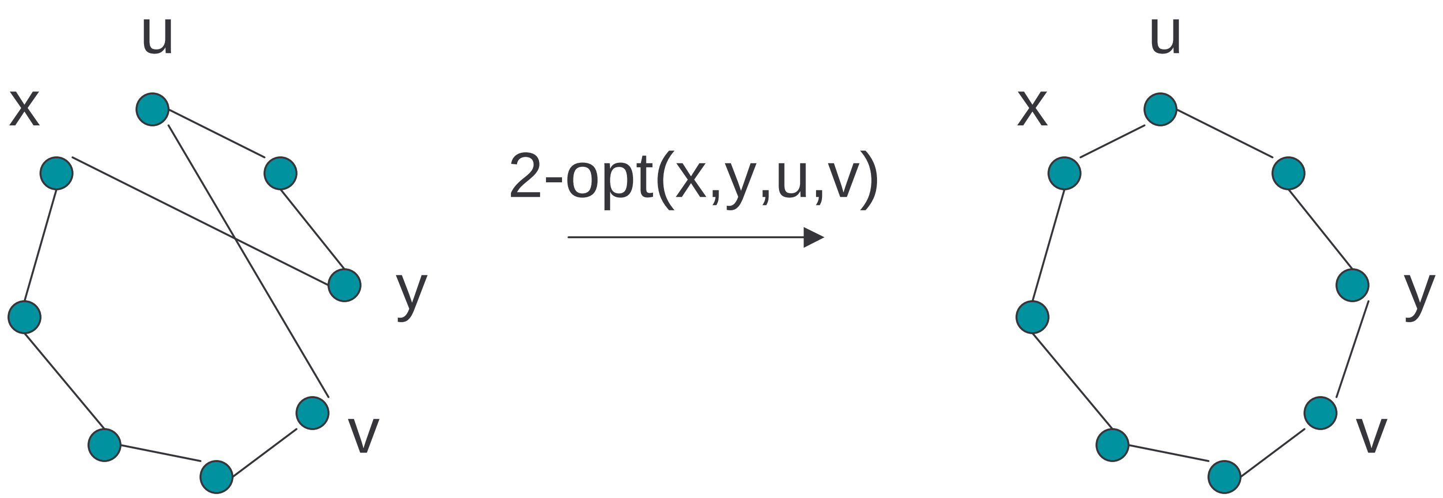
Solving TSP Approximately, Practically

- Local Search
  - Lin-Kernighan method
- Simulated Annealing
- Genetic Algorithms
- Neural Networks

Local Search Algorithms

- Start with an initial solution that is usually easy to find, but is not necessarily good.
- Repeatedly modify the current solution to a better nearby one. Until no nearby one is better.

2-Opt Neighborhood



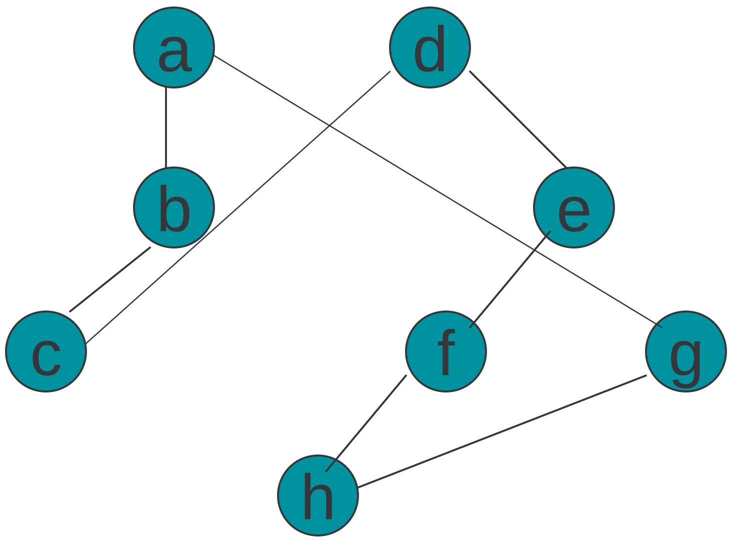
2-opt Algorithm

Lin-Kernighan (1973)

```
    Find an initial tour T
1. For every pair of distinct edges (x,y), (u,v) in T
      if C(x,u) + C(y,v) < C(x,y) + C(u,v) then
        T := T – {(x,y),(u,v)} union {(x,u),(y,v)}
        exit for loop and go to 1
    Return T
```

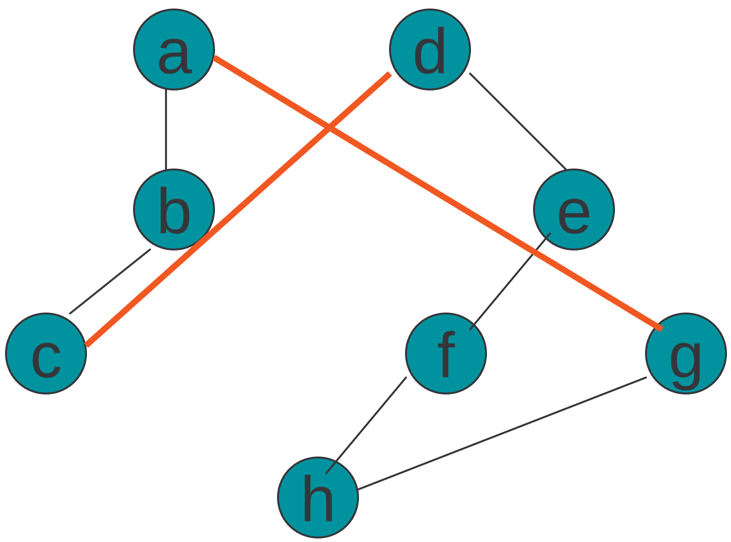
Example of LK

Euclidian case



Example of LK

Euclidian case



Example of LK

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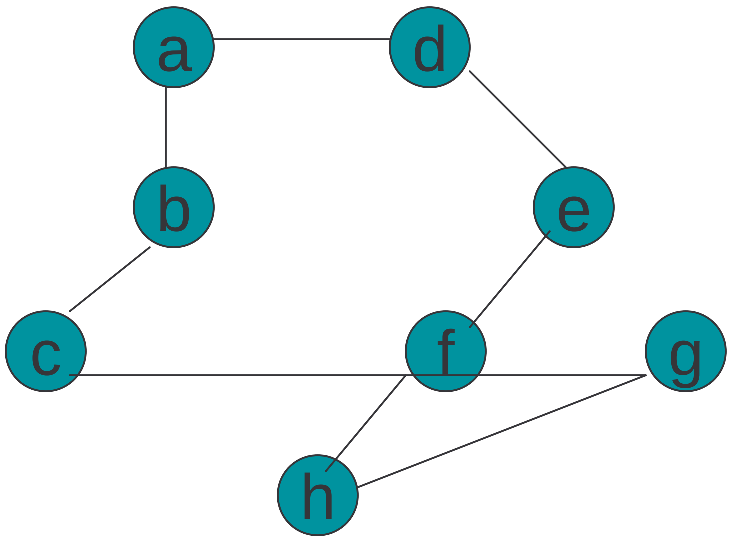

Example of LK

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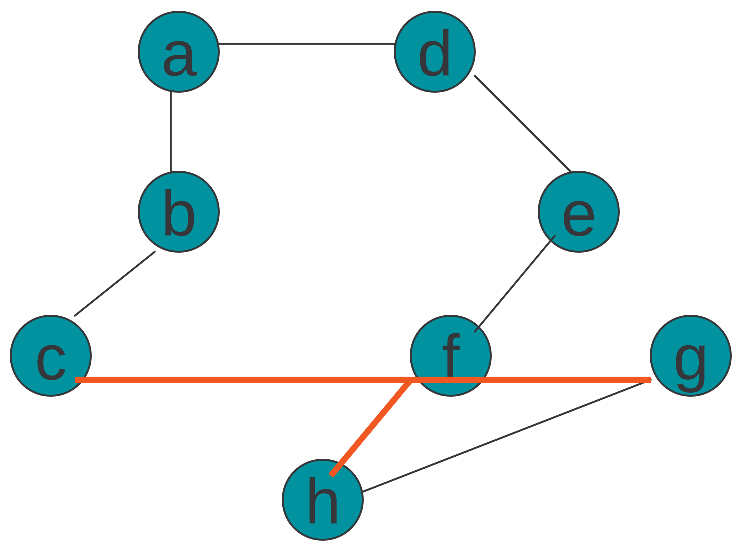
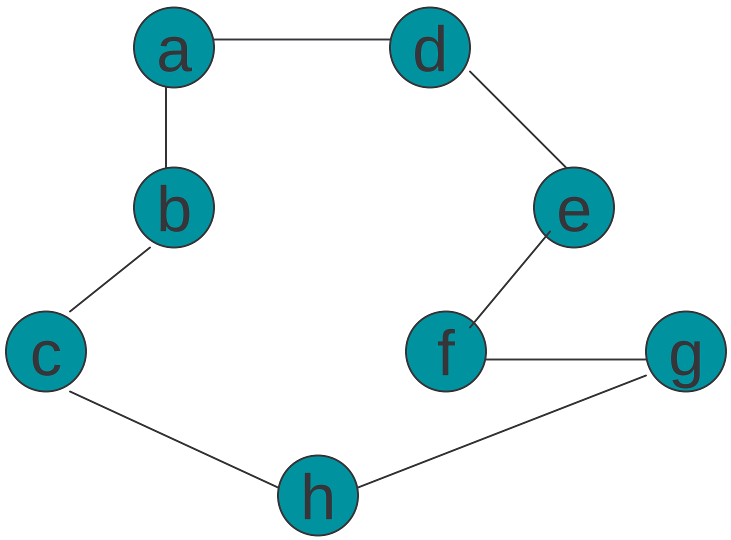


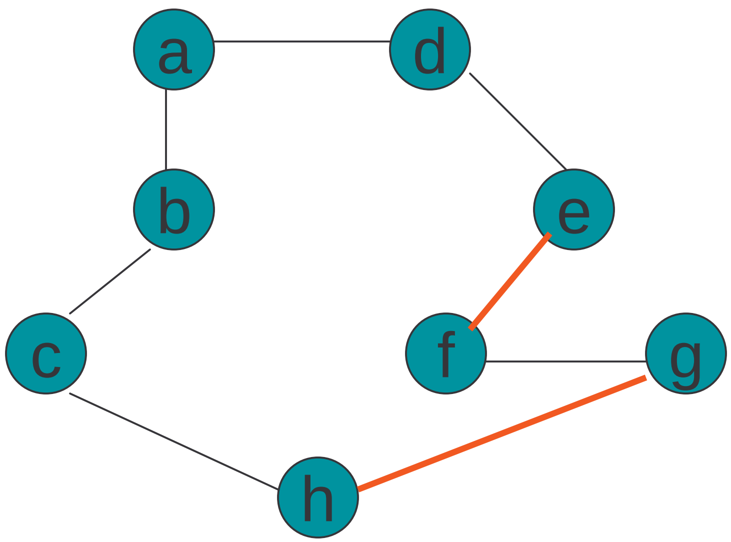
Example of LK

Euclidian case



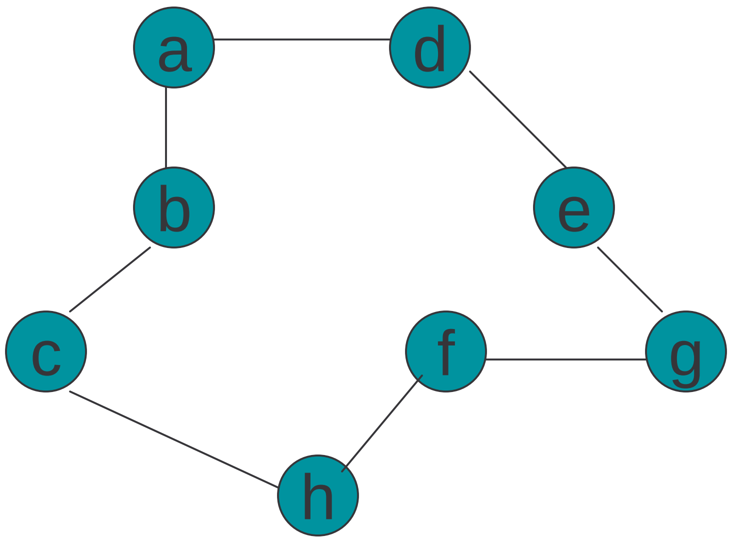
Example of LK

Euclidian case



Example of LK

Euclidian case

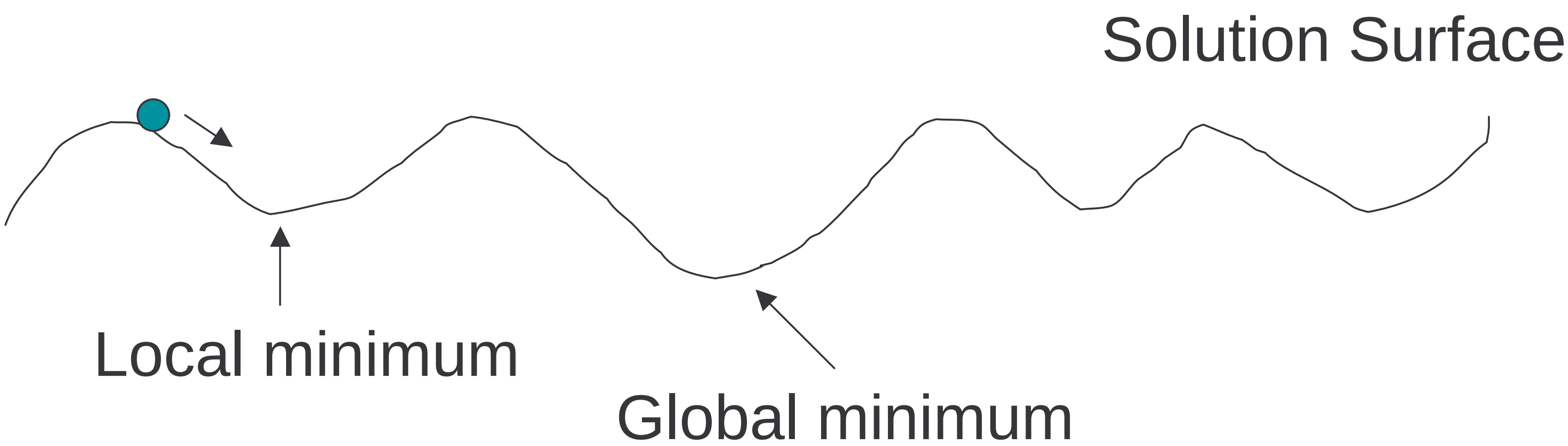


Lin-Kernighan

- Empirical $O(n^{2.2})$ time
- Finds optimal in most examples < 100 points
- Excellent Implementations
  - Can easily handle hundreds of thousands of points

Local Minimum Problem

- Local search can lead to a local minimum in the solution space, not necessarily a global minimum.



NP-Completeness Theory

- Explains why some problems are hard and probably not solvable in polynomial time.
- Invented by Cook in 1971.
- Popularized in an important paper by Karp in 1972.
- Standardized by Garey and Johnson in 1979 in "Computers and Intractability: A Guide to the Theory of NP-Completeness".

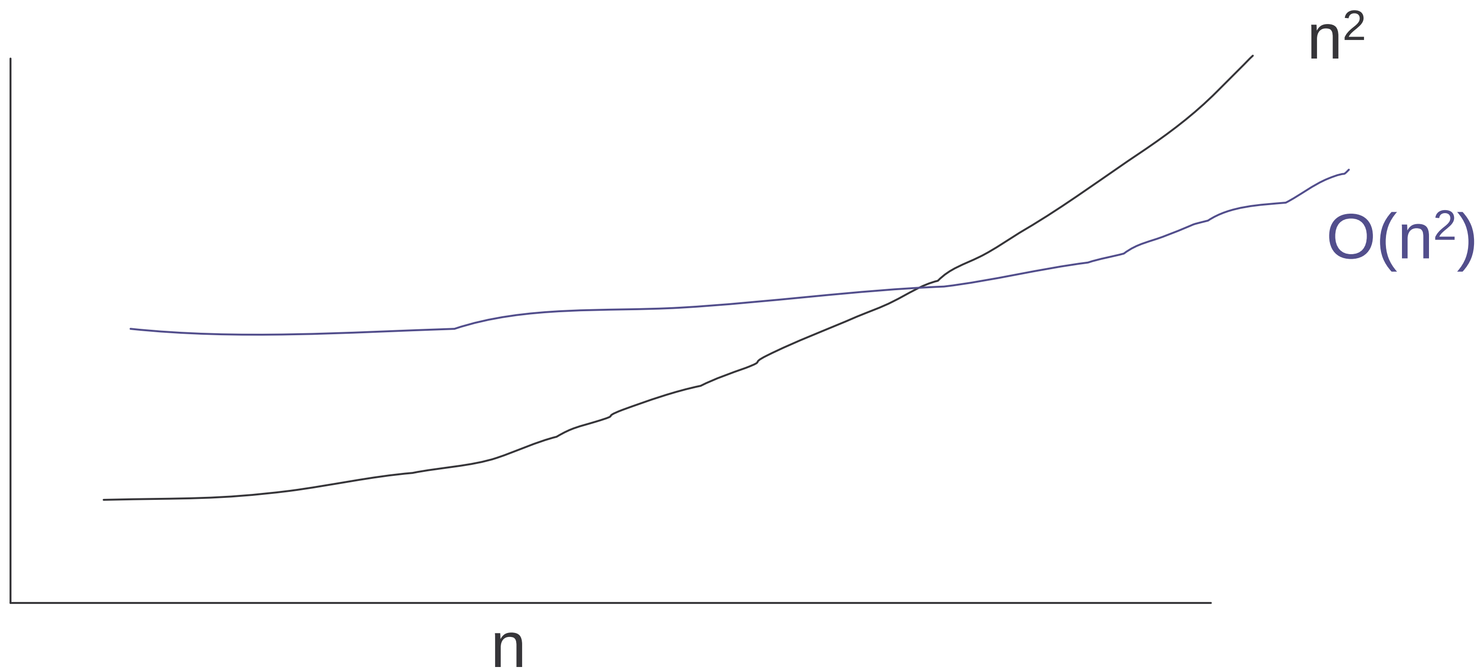
P

- Complexity theory is the study of the time and storage needed to solve problems.
  - Sorting requires $\Theta(n \log n)$ time
  - Minimum spanning tree can be solved in $O(m \log m)$ time
  - Connected components can be solved in $O(m)$ time.
- P is the class of problems that can be solved in polynomial time.
  - $O(n)$, $O(n^2)$, $O(n^3)$, ... time

Order Notation

- $f(n) = O(g(n))$ means $f(n) \leq c\, g(n)$ for some c.
  - $1{,}000{,}000\, n^2 + 2n = O(n^2)$
  - $n \log n = O(n^3)$
- $f(n) = \Omega(g(n))$ means $f(n) \geq c\, g(n)$ for some $c > 0$.
  - $.0000001\, n^2 + 2n = \Omega(n^2)$
  - $1{,}000\, n^2 = \Omega(n)$
- $f(n) = \Theta(g(n))$ means $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
  - $a_k n^k + a_{k-1} n^{k-1} + \ldots = \Theta(n^k)$ if $a_k > 0$

Graph of Order of Magnitude



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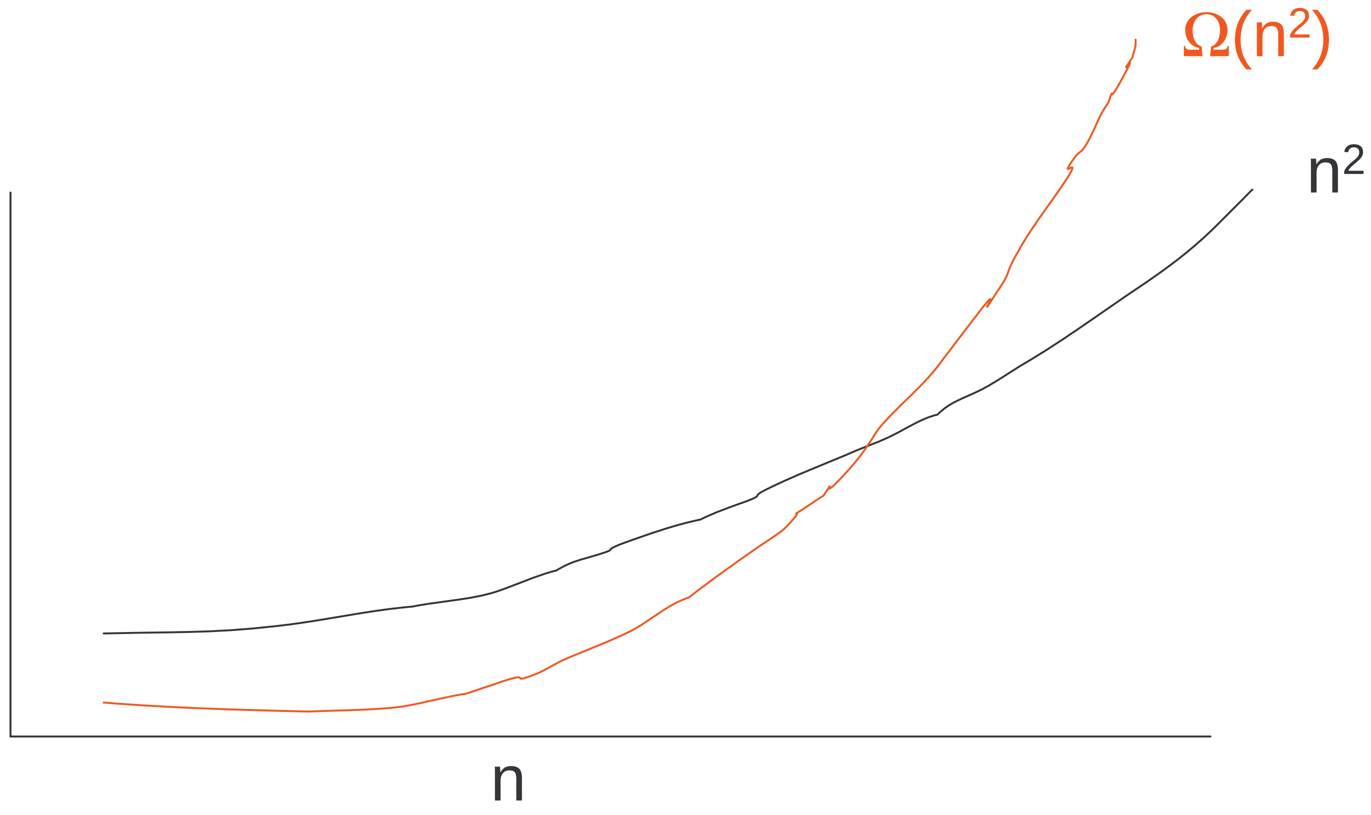
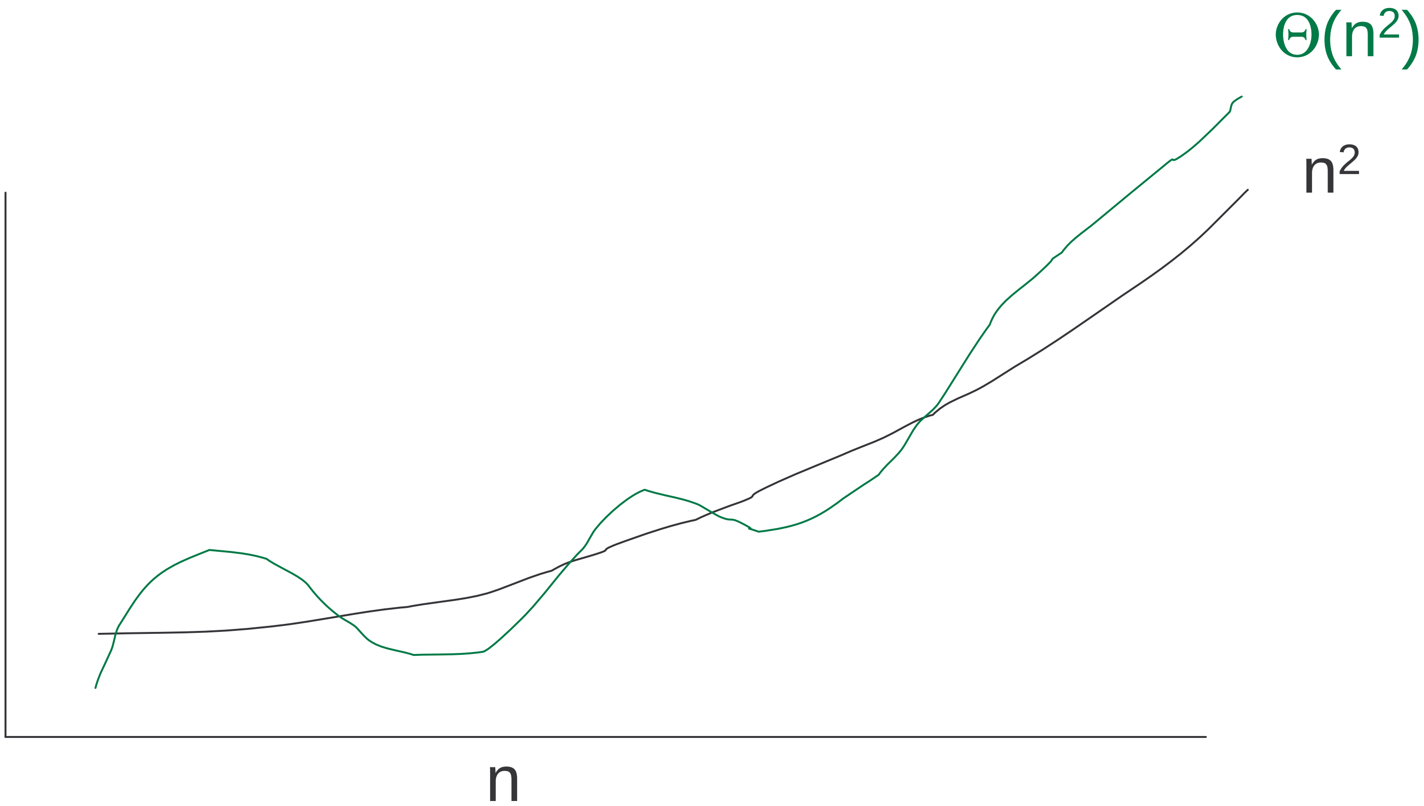


Graph of Order of Magnitude

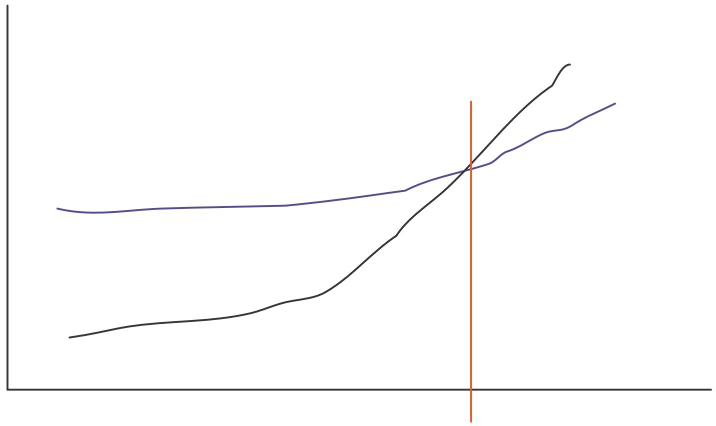


Worst Case Asymptotic Analysis

- Given problem find the best t(n) such that there is an algorithm solving the problem that runs in time O(t(n)) on all inputs of size n.
  - t(n) is an asymptotic upper bound
- Given a problem find the best t'(n) such that every algorithm solving the problem runs in time Ω(t'(n)) on some input of length n.
  - t'(n) is an asymptotic lower bound

Bane of Worst Case Asymptotic Analysis

- Worst case
  - A bad asymptotic algorithm in the worst case might do well on the common case.
- Asymptotic
  - A good asymptotic algorithm might perform poorly on inputs of reasonable size.



crossover is large

NP

- NP stands for nondeterministic polynomial time.
- We consider the class of decision problems (yes/no problems).
- A nondeterministic algorithm is one that can make “guesses”.
- A decision problem is in NP if it can be solved by a nondeterministic algorithm that runs in polynomial time.
- Some problems in NP seem very hard to solve.

Examples of Decision Problems in NP

- Decision TSP
  - Input: Graph G = (V,E) with costs on the edges and a budget B
  - Output: Determine if there is a tour visiting each vertex exactly once of total cost $\leq$ B.
  - Algorithm: Guess a tour and check its cost is under budget.
- Graph Coloring
  - input: Graph G = (V,E) and a number k.
  - output: Determine if all vertices can be colored with k colors such that no two adjacent vertices have the same color.
  - Algorithm: Guess a coloring and then check it.

CNF-SAT

- Input: A Boolean formula F in conjunctive normal form.

$$(x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

- Output: Determine if F is satisfiable, that is, there is some assignment to the variables that makes the formula F true.

$$x = 1,\ y = 0,\ z = 1$$

$$(1 \vee 0 \vee 1) \wedge (\neg 1 \vee 0 \vee 1) \wedge (\neg 1 \vee \neg 0 \vee \neg 1)$$

- Algorithm: Guess an assignment and check it.

Subset Sum

- Input: Integers $a_1, a_2, \ldots, a_n, b$
- Output: Determine if there is subset

$$X \subseteq \{1,2,\ldots,n\}$$

with the property $\sum_{i \in X} a_i = b$

- Algorithm: Guess the subset X and check the sum adds up to b.

Decision Problems
Reporting Problems
Optimization Problems

- Example 1: Subset sum
  - Decision Problem: Determine if a subset sum exists.
  - Reporting Problem: If a subset sum exists, then report one.
  - Optimization Problem: Find a subset whose sum is as close as possible to b, without going over b.

Decision Problems
Reporting Problems
Optimization Problems

- Example 2. Traveling Salesman
  - Optimization problem – Find a tour that minimizes cost.
  - Decision problem – Determine if a tour exists that comes under a specified budget.
  - Reporting problem - If a tour exist that comes under a specified budget, find it.

Polynomial Time Equivalence of Decision, Reporting, Optimization

- If any one of Decision, Reporting, or Optimization can be solved in polynomial time then so can the others.
- Decision is easily reducible to Optimization
  - Subset sum
  - Traveling salesman

Reporting Reduces to Decision

- Subset sum:
  - Let subset-sum(A,b) return true if some subset of A adds up to b. Otherwise it returns false.

```
Precondition: subset-sum ({a_1,…,a_n},b) is true
Report ({a_1,…,a_n},b)
X := the empty set;
for i = 1 to n do
    if subset-sum({a_{i+1},…,a_n},b - a_i) then
        add i to X;
        b := b - a_i;
```

Example

3, 5, 2, 7, 4, 2, b = 11

5, 2, 7, 4, 2, b = 11-3 --> yes, X = {3}, b = 8

2, 7, 4, 2, b = 8-5 --> no

7, 4, 2, b = 8-2 --> yes, X = {3,2}, b = 6

4, 2, b = 6-7 --> no

2, b = 6-4 --> yes, X = {3,2,4}, b = 2

b = 2 -2 --> yes, X ={3,2,4,2}

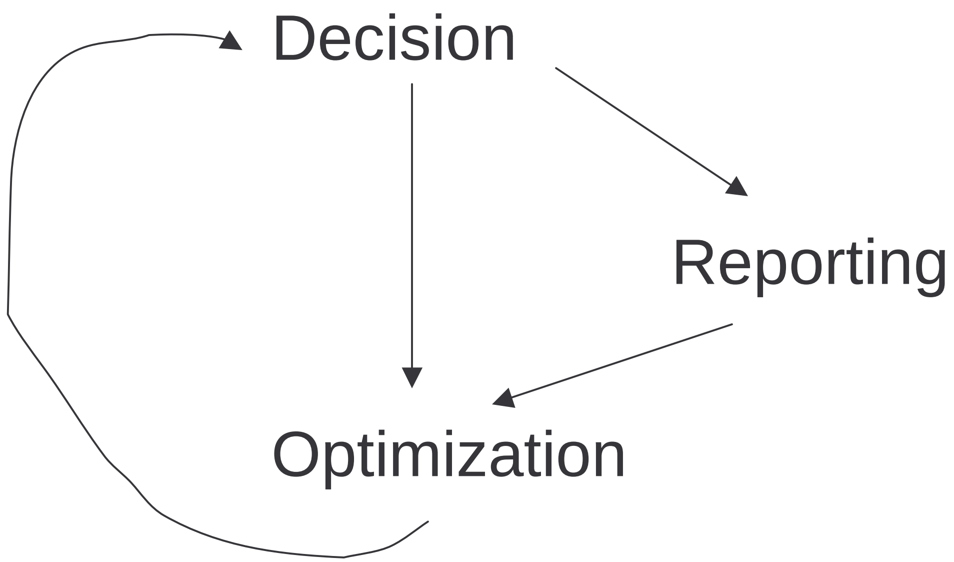
Optimization Reduces to Decision

- Traveling Salesman
  - TS(G,B) which returns true if and only if G has a tour of length ≤ B. Assume costs are positive integers.

1. Find the minimum cost of a tour by binary search
2. Find the tour itself (reporting).

```
Find minimum cost of a tour
L := 0;
U := sum of all costs of edges;
while L + 1 < U do
   B = (L+U)/2;
   if TB(G,B) then U := B else L := B;
return U
```

The Relationship



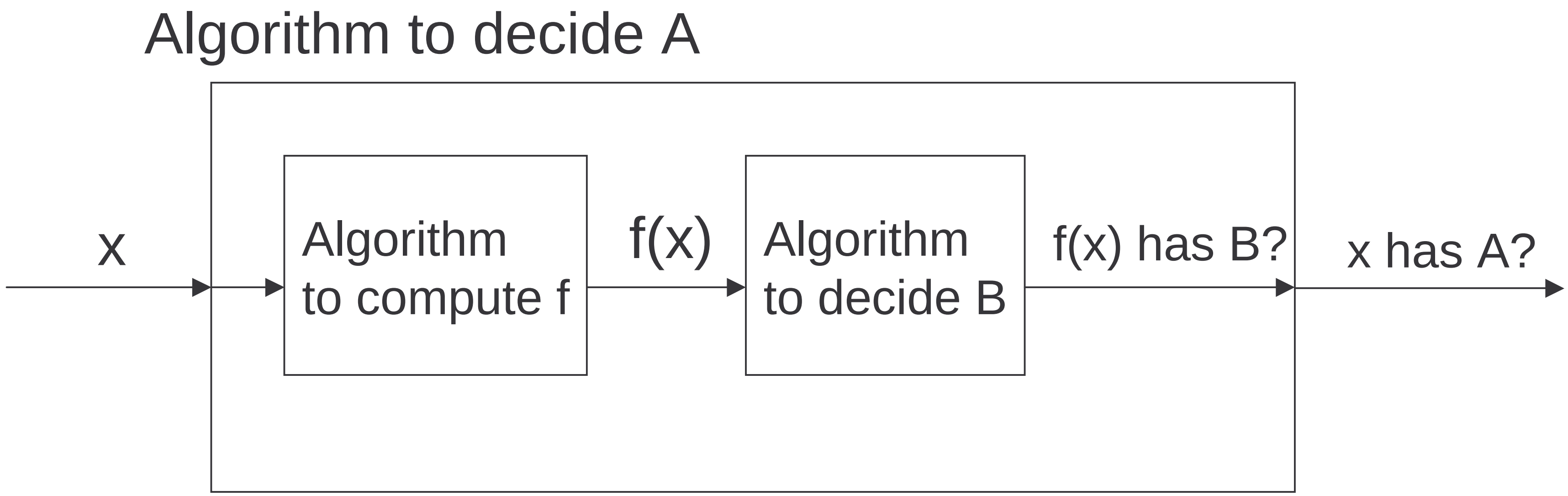
Exercise

1. Assume the decision algorithm subset-sum(A,b) is provided. Solve the optimization problem for subset sum.
2. Assume the decision problem TS(G,B) is given. Solve the reporting problem for traveling salesman.

Polynomial Time Reducibility

- Informal idea: A decision problem A is polynomial time reducible to a decision problem B if a polynomial time algorithm for B can be used to construct a polynomial time algorithm for A.
- Formally: A is polynomial time reducible to B if there is a function f computable in polynomial time such that for all x:
  - x has A if and only if $f(x)$ has B
- If A polynomial time reducible to B and B solvable in polynomial time then so is A.

Block Diagram to Decide A from B



Transitivity of Polynomial Time Reduction

- Define: $A \leq_P B$ to mean that A is polynomial time reducible to B.
- Transitivity: $A \leq_P B$ and $B \leq_P C$ implies $A \leq_P C$
- Example:
  - Every problem in NP is known to be polynomial time reducible to CNF-SAT.
  - SAT is polynomial time reducible to Decision TSP
  - Therefore, every problem in NP is polynomial time reducible to Decision TSP.

NP-Completeness Definition

- Definition: A decision problem A is NP-complete if
  - A is in NP
  - Every problem in NP is reducible to A in polynomial time.

- NP-complete problems seem to require exponential time, but there is no proof to date.

Cook’s Theorem

- CNF-satisfiability is NP-complete
  - Cook 1971, Levin 1973

Proof formalizes the notion of a nondeterministic algorithm as a nondeterministic Turing machine. It can be shown that a CNF-formula F can be produced in polynomial time that describes the operation of the nondeterministic Turning machine. The Turing machine halts in a “yes” state if and only if the formula F is satisfiable.

NP-Hardness

- Definition: A problem A is NP-hard if an NP-complete problem can be solved using A as an “oracle”.
  - Decision TSP is NP-complete
  - TSP is NP-hard
- Oracle is like a constant time function call.

P vs NP

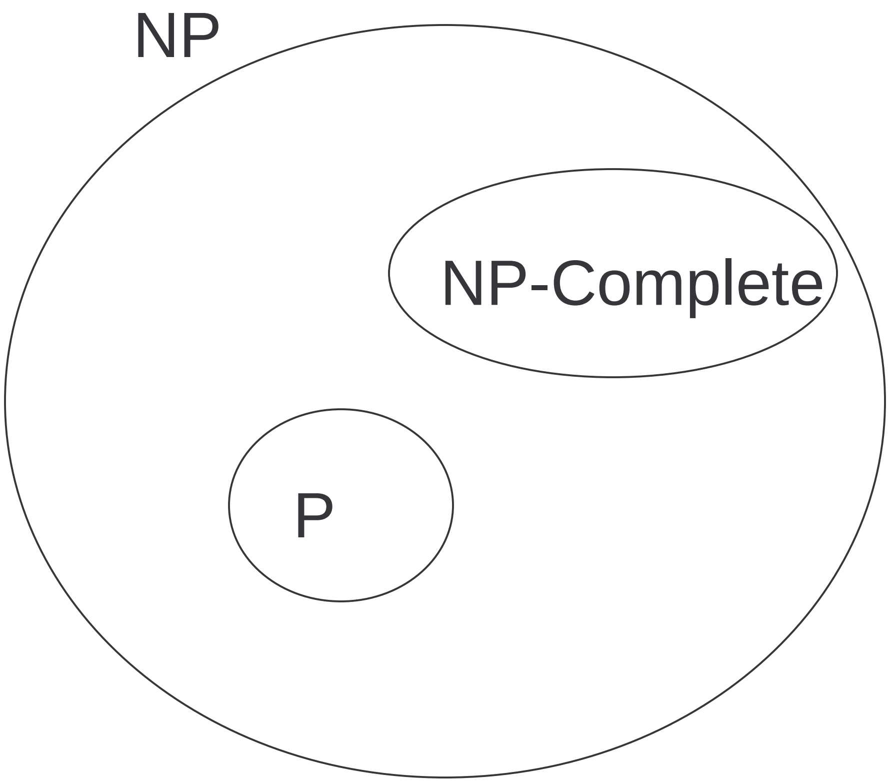
- Every problem in P is also in NP

$$P \subseteq NP$$

- Famous UnsolvedOpen Question:

$$P = NP\,?$$

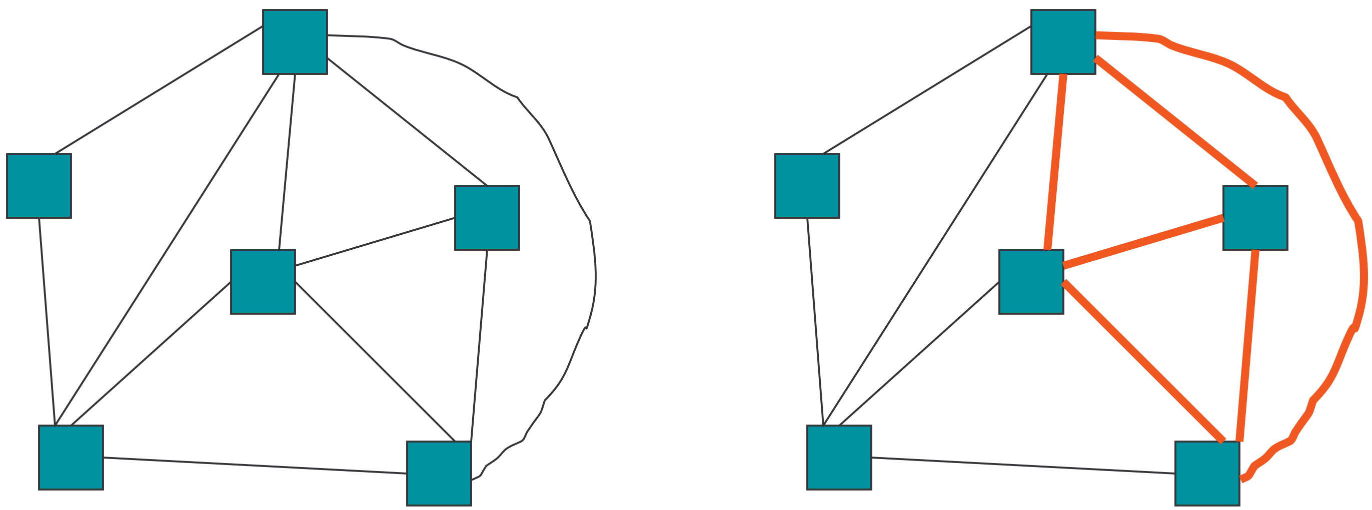
Probable Picture



Clique Decision Problem

- Input: Undirected Graph G = (V,E) and a number k.
- Output: Determine if G has a k-clique, that is, there is a set of vertices U of size k such that for each pair of vertices in U there is and edge in E between them.

Clique Example



4-clique

Clique is NP-Complete

- Clique is in NP
  - Nondeterministic algorithm: guess k vertices then check that there is an edge between each pair of them.
- Clique is NP-hard
  - We reduce CNF-satisfiability to Clique in polynomial time
  - Given a CNF formula F we need to construct a graph G and a number k with the property that F is satisfiable if and only if G has a k-clique. The contstruction must be efficient, polynomial time.

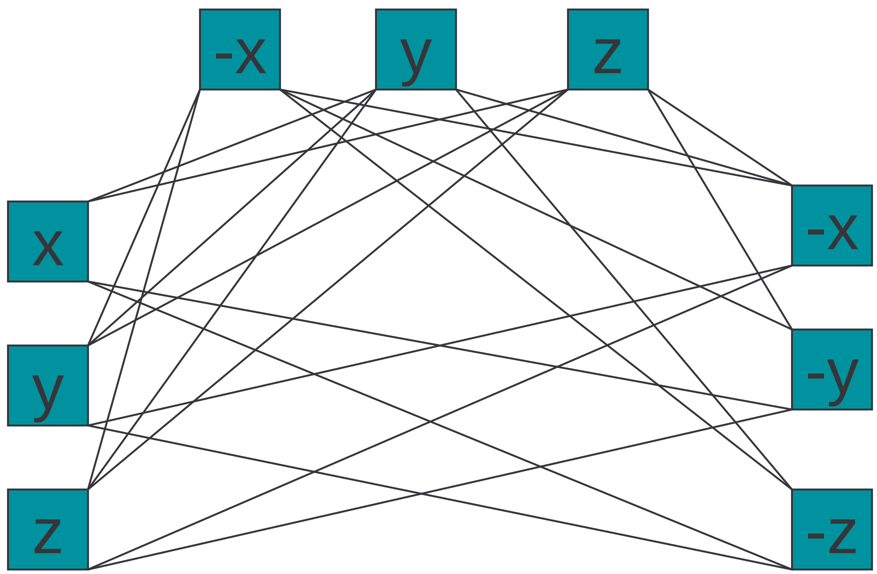
Construction by Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

literal

clause

G

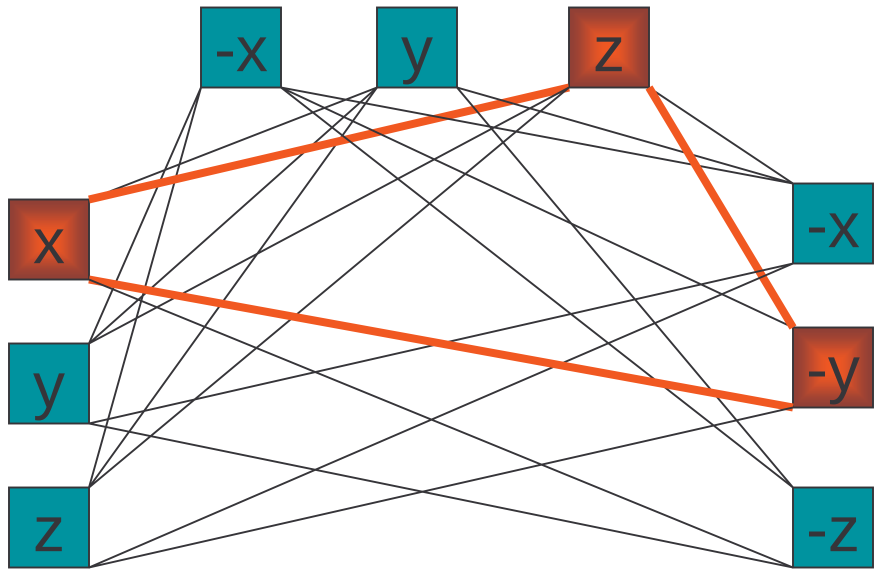


Construction by Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

$$x = 1,\ y = 0,\ z = 1$$

G



General Construction

$$F = \bigcap_{i=1}^{k} \bigcup_{j=1}^{m_i} a_{ij}$$ where $a_{ij} \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$

literals

$$G = (V, E)$$ where

$$V = \{a_{ij} : 1 \le i \le k, 1 \le j \le m_i\}$$

$$E = \{\{a_{ij}, a_{i'j'}\} : i \ne i' \, and,$$

$$a_{ij} \text{ and } a_{i'j'} \text{ are not complementary}\}$$

k *is the number of clauses*

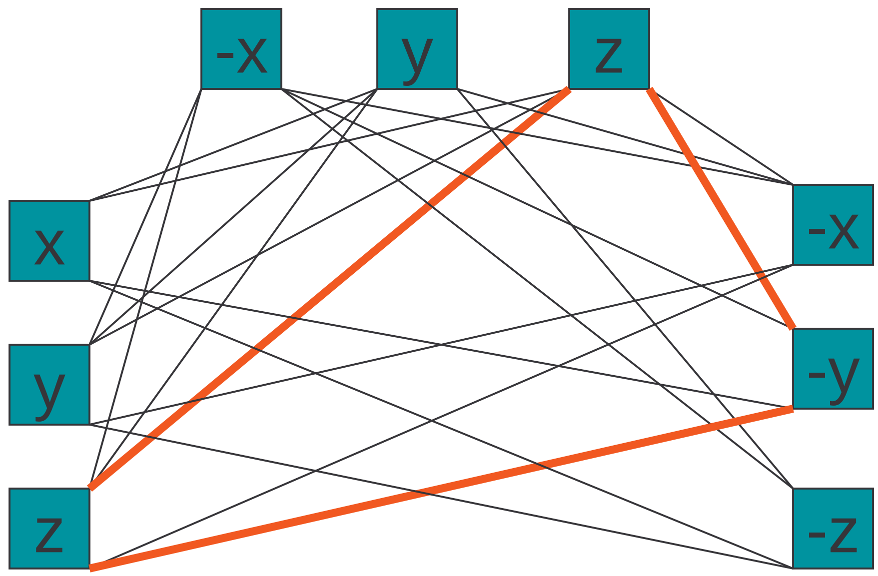
The Reduction Argument

- We must show
  - F satisfiable implies G has a clique of size k.
    - Given a satisfying assignment for F, for each clause pick a literal that is satisfied. Those literals in the graph G form a k-clique.
  - G has a clique of size k implies F is satisfiable.
    - Given a k-clique in G, assign each literal in the clique to be 1. This yields a satisfying assignment to F.

Clique to Assignment

$$F = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

G

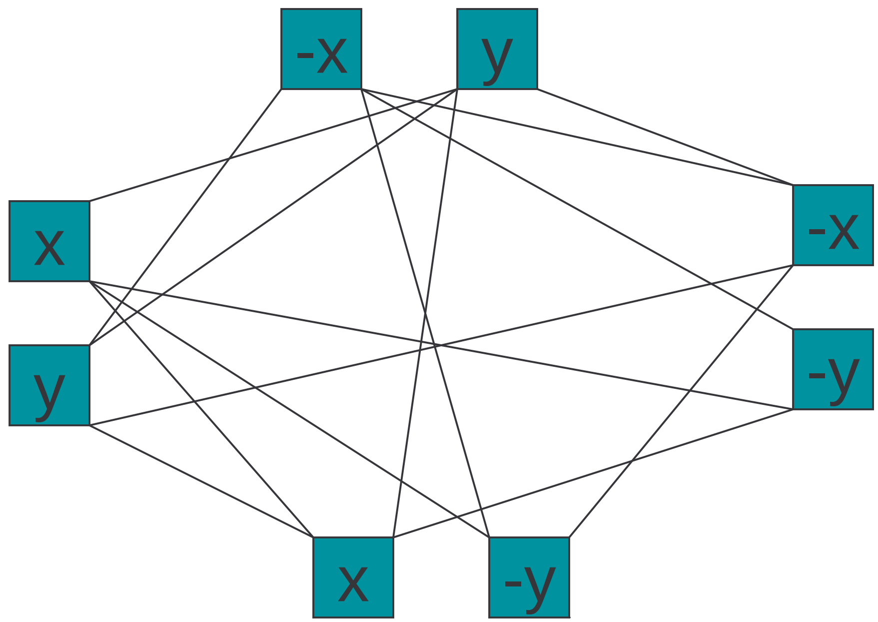


$$y = 0,\ z = 1$$

Assignment to Clique

$$F = (x \vee y) \wedge (\neg x \vee y) \wedge (\neg x \vee \neg y) \wedge (x \vee \neg y)$$

G



G has no 4-clique

3-CNF-Satifiability

- Input: A Boolean formula F with at most 3 literals per clause.
- Output: Determine if F is satisfiable.

- 3-CNF-Satisfiability is NP-complete
  - This is probably the most used NP-complete problem in reduction proofs showing other decision problems are NP-hard.

Reduction by Example

Given $F = (x_1 \vee \neg x_2 \vee x_3 \vee \neg x_4) \wedge F'$

Construct $H = (x_1 \vee z_1) \wedge (\neg x_2 \vee \neg z_1 \vee z_2)$
$\wedge (x_3 \vee \neg z_2 \vee z_3) \wedge (\neg x_4 \vee \neg z_3) \wedge F'$

F is satisfiable if and only if H is satisfiable.

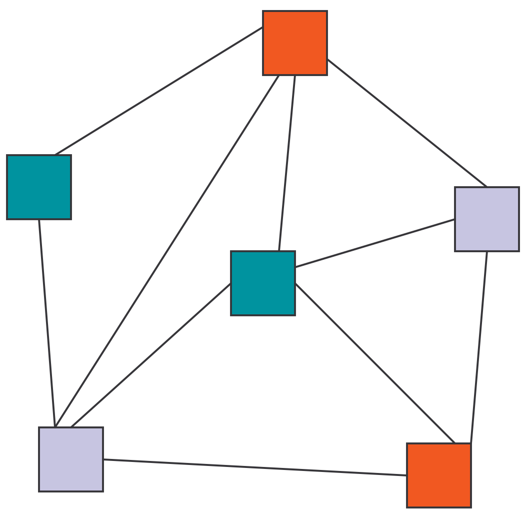
$x_2 = 0$ satisfies the first clause of F.

$z_1 = 1,\ z_2 = 0,\ z_3 = 0$ satisfy clauses 1,3, and 4 of H and

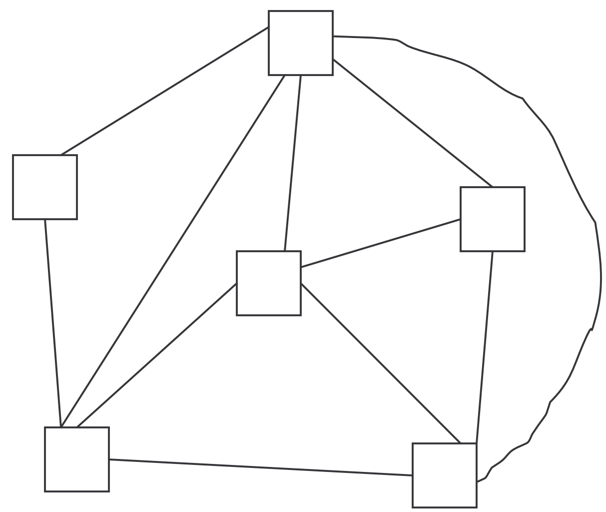
$x_2 = 0$ satisfies the clause 2 of H.

3-Colorability

- Input: Graph G = (V,E).
- Output: Determine if all vertices can be colored with 3 colors such that no two adjacent vertices have the same color.



3-colorable



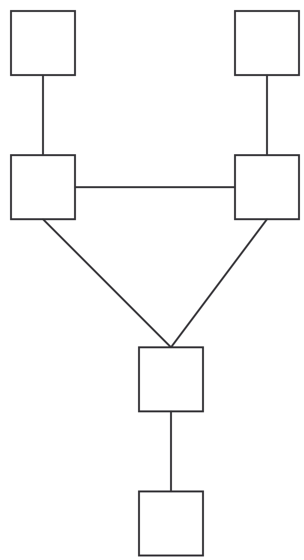
Not 3-colorable

3-CNF-Sat $\leq_P$ 3-Color

- Given a 3-CNF formula *F* we have to show how to construct in polynomial time a graph *G* such that:
  - *F* is satisfiable implies *G* is 3-colorable
  - *G* is 3-colorable implies *F* is satisfiable

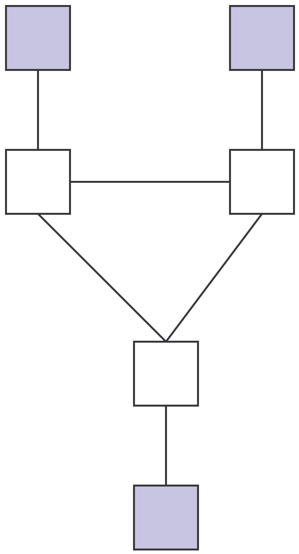
The Gadget

- This is a classic reduction that uses a “gadget”.
- Assume the outer vertices are colored at most two colors. The gadget is 3-colorable if and only if the outer vertices are not all the same color.

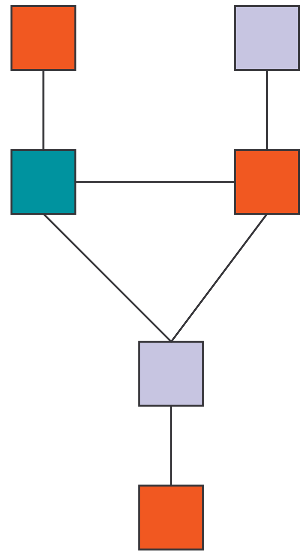


Properties of the Gadget

- Three colorable if and only if outer vertices not all the same color.



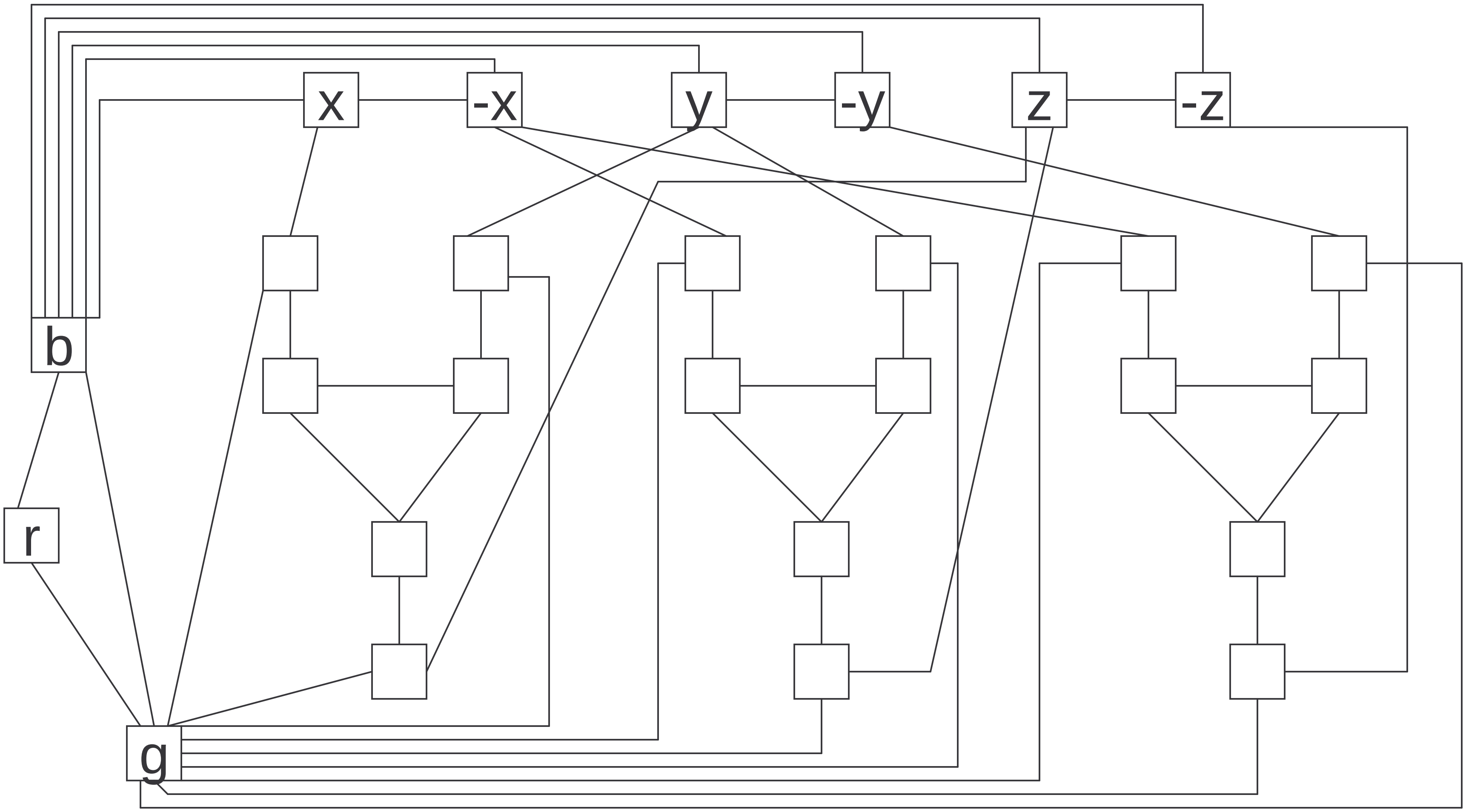
Not 3 colorable



Is 3 colorable

Reduction by Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$



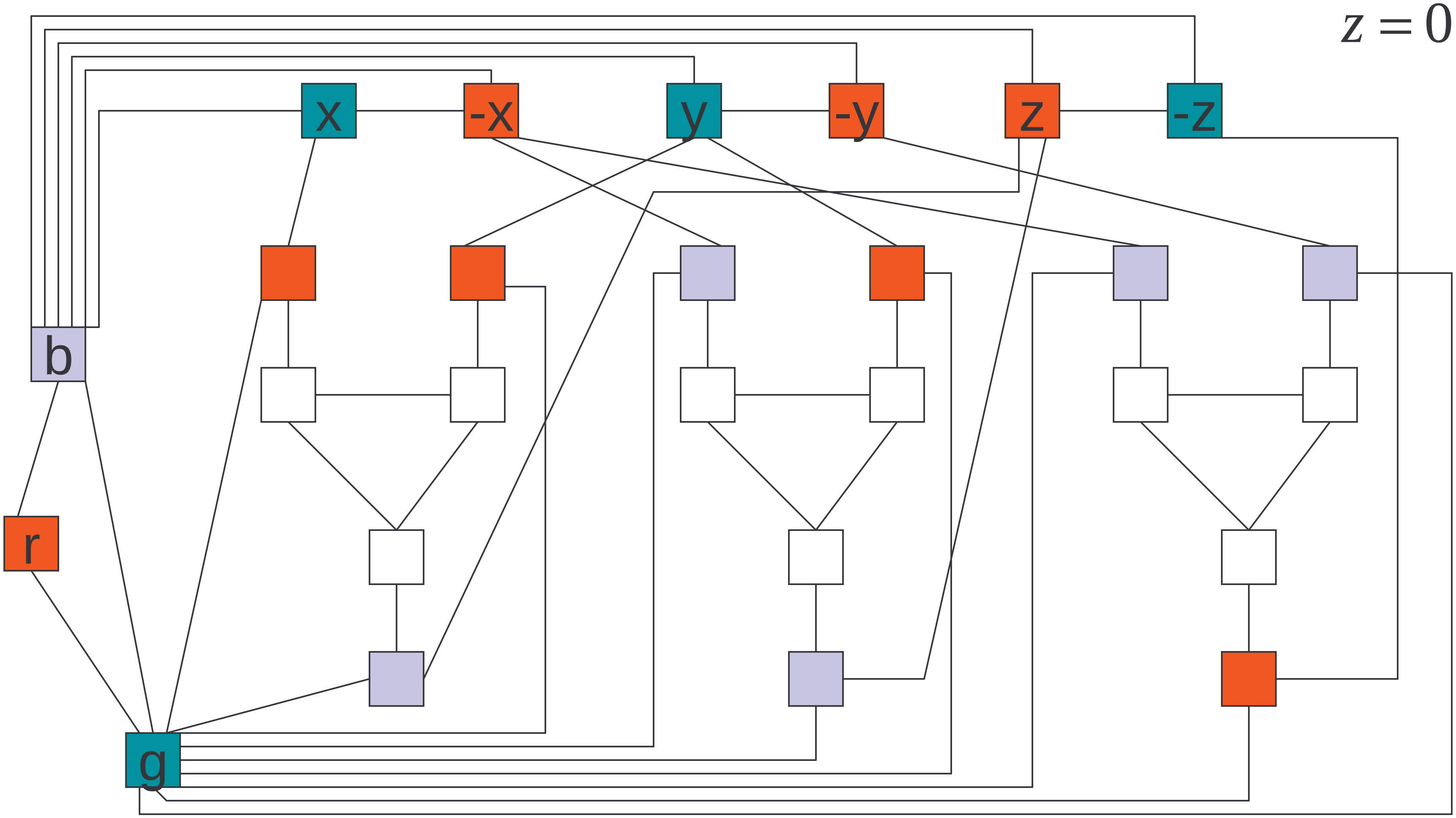
Satisfaction Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

$x = 1$

$y = 1$

$z = 0$



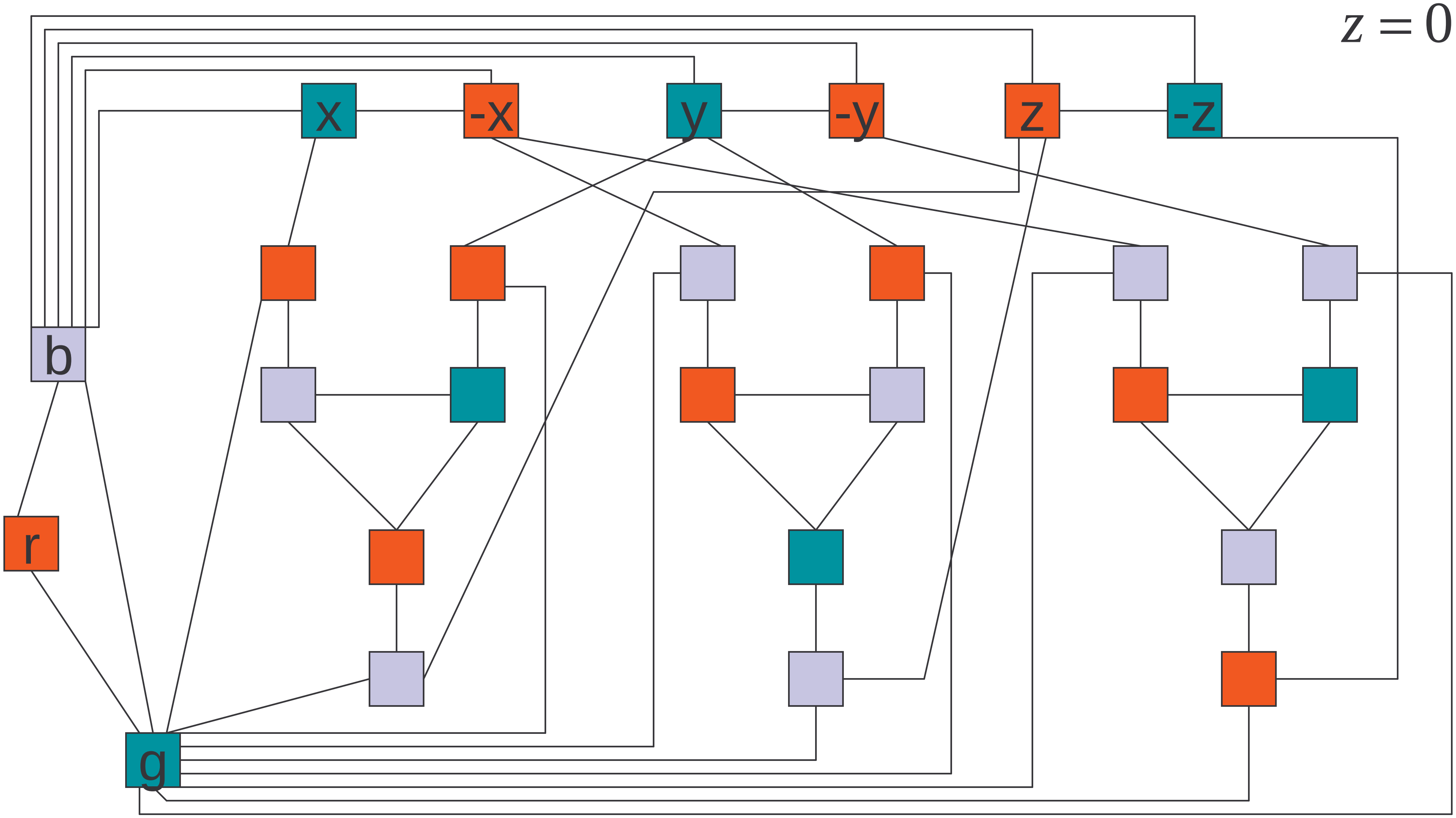
Satisfaction Example

$$F = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

$x = 1$

$y = 1$

$z = 0$



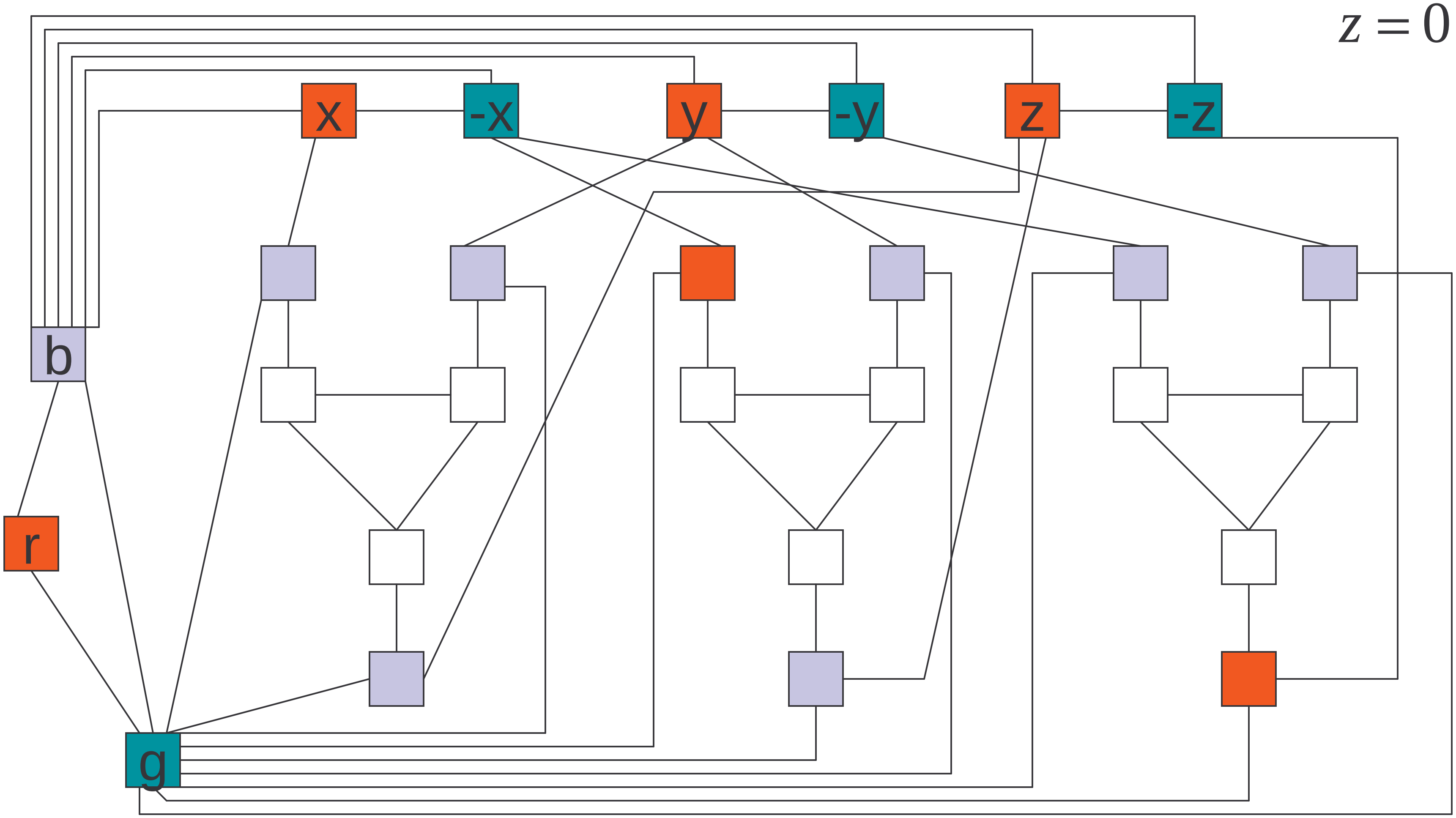
Non-Satisfaction Example

$x = 0$

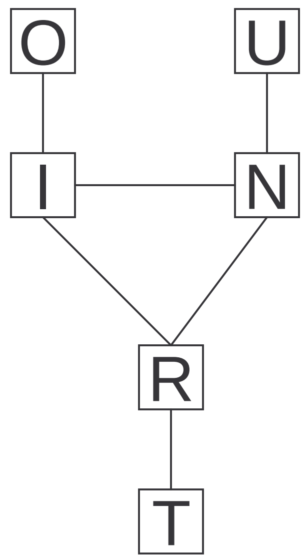
$$F = (x \vee y \vee z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

$y = 0$

$z = 0$



Naming the Gadget



General Construction

$$F = \bigcap_{i=1}^{k} (a_{i1} \vee a_{i2} \vee a_{i3}) \quad \text{where} \quad a_{ij} \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$$

$$G = (V, E) \qquad \text{where}$$

$$V = \{r, g, b\} \cup \{x_1, \neg x_1, \ldots, x_n, \neg x_n\} \cup \{O_i, U_i, T_i, I_i, N_i, R_i : 1 \le i \le k\}$$

$$E = \{\{r, g\}, \{g, b\}, \{b, r\}\}$$

$$\cup \{\{x_1, \neg x_1\}, \ldots, \{x_n, \neg x_n\}\}$$

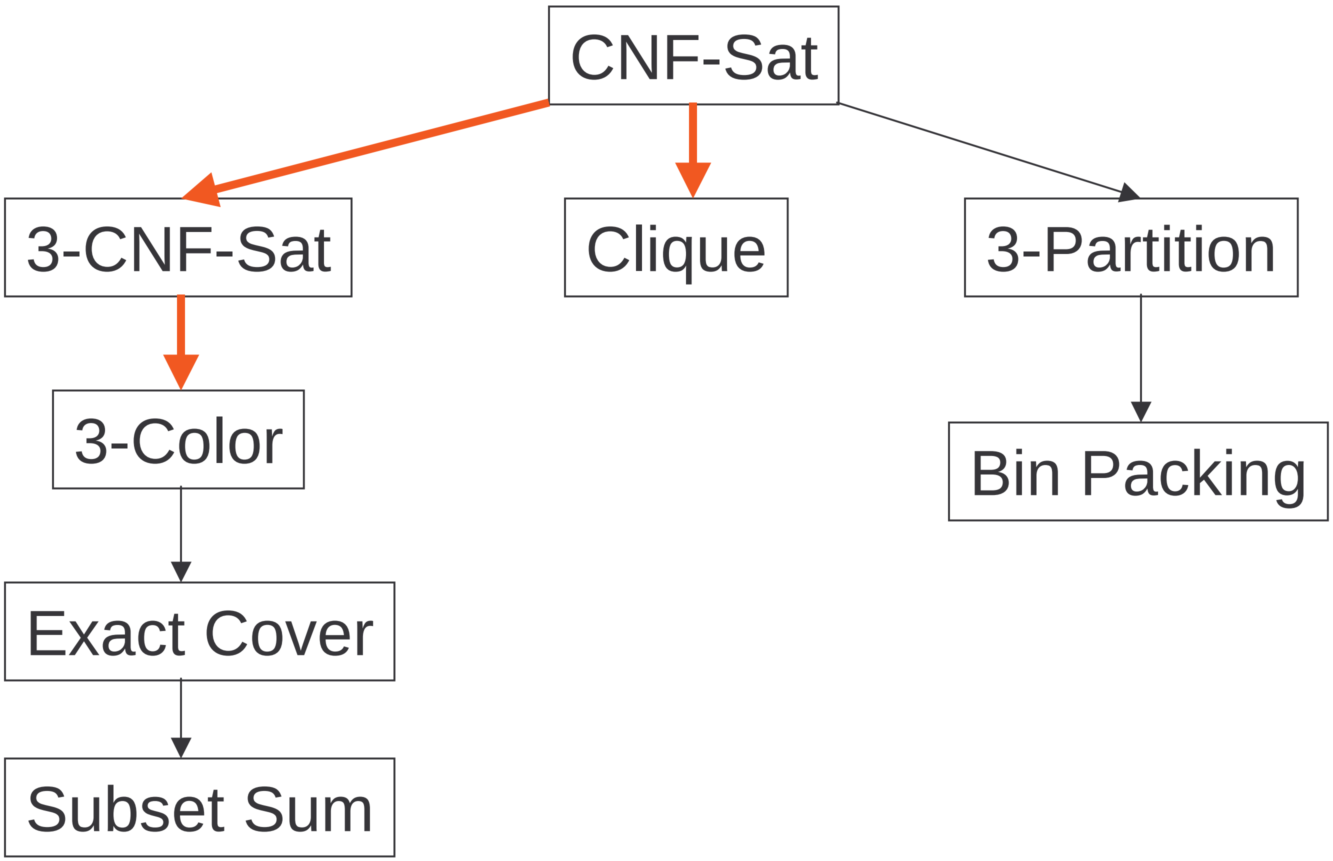
$$\cup \{\{x_1, b\}, \{\neg x_1, b\}, \ldots, \{x_n, b\}, \{\neg x_n, b\}\}$$

$$\cup \{\{O_i, I_i\}, \{U_i, N_i\}, \{T_i, R_i\}, \{I_i, N_i\}, \{N_i, R_i\}, \{R_i, I_i\} : 1 \le i \le k\}$$

$$\cup \{\{a_{i1}, O_i\}, \{a_{i2}, U_i\}, \{a_{i3}, T_i\} : 1 \le i \le k\}$$

$$\cup \{\{O_i, g\}, \{U_i, g\}, \{T_i, g\} : 1 \le i \le k\}$$

Reductions


CNF-Sat
3-CNF-Sat
Clique
3-Partition
3-Color
Bin Packing
Exact Cover
Subset Sum

Exact Cover

- Input: A set $U = \{u_1, u_2, \ldots, u_n\}$ and subsets

$$S_1, S_2, \ldots, S_m \subseteq U$$

- Output: Determine if there is set of pairwise disjoint sets that union to U, that is, a set X such that:

$$X \subseteq \{1,2,\ldots,m\}$$

$$i, j \in X \text{ and } i \neq j \text{ implies } S_i \cap S_j = \phi$$

$$\bigcup_{i \in X} S_i = U$$

Example of Exact Cover

$$U = \{a, b, c, d, e, f, g, h, i\}$$

$$\{a, c, e\}, \{a, f, g\}, \{b, d\}, \{b, f, h\}, \{e, h, i\}, \{f, h, i\}, \{d, g, i\}$$

Exact Cover

$$\{a, c, e\}, \{b, f, h\}, \{d, g, i\}$$

3-Partition

- Input: A set of numbers $A = \{a_1, a_2, \ldots, a_{3m}\}$ and number B with the properties that $B/4 < a_i < B/2$ and
$$\sum_{i=1}^{3m} a_i = mB.$$
- Output: Determine if A can be partitioned into S_1, $S_2, \ldots, S_m$ such that for all i
$$\sum_{j \in S_i} a_j = B.$$

Note: each S_i must contain exactly 3 elements.

Example of 3-Partition

- A = {26, 29, 33, 33, 33, 34, 35, 36, 41}
- B = 100, m = 3
- 3-Partition
  - 26, 33, 41
  - 29, 36, 35
  - 33, 33, 34

Bin Packing

- Input: A set of numbers $A = \{a_1, a_2, \ldots, a_m\}$ and numbers B (capacity) and K (number of bins).
- Output: Determine if A can be partitioned into S_1, S_2,…, S_K such that for all i

$$\sum_{j \in S_i} a_j \leq B.$$

Bin Packing Example

- A = {2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5}
- B = 10, K = 4
- Bin Packing
  - 3, 3, 4
  - 2, 3, 5 Perfect fit!
  - 5, 5
  - 2, 4, 4

Exercise – Argue NP-Completeness

1. Independent Set
   – Input: Undirected graph G = (V,E) and a number k.
   – Output: Determine if there is an independent set of size k. X, contained in V, is independent if for all i,j in X there is no edge in G from i to j.
2. Equal Subset-Sum
   – Input: $\{a_1, a_2, \ldots, a_n\}$ positive integers
   – Output: Determine if there is a set I such that

$$\sum_{i \in I} a_i = \sum_{j \notin I} a_j$$

Coping with NP-completeness

- You have encountered a Hard Problem
- Maybe it is NP-hard
  - Books
    - Garey and Johnson
  - Websites
    - http://www.nada.kth.se/~viggo/problemlist/compendium.html
  - Research papers
  - Maybe you'll have to do your own reduction
- Can't determine NP-hardness, then it is probably hard in some way.
- Modify the problem to be more tractable

Boundary Between P and NP

- Satisfiability
  - 2-CNF-SAT is in P
  - 3-CNF-SAT is NP-complete
- Coloring
  - 2-COLOR is in P
  - 3-COLOR is NP-complete
- Planar Colorability
  - Planar graphs are always 4-colorable
  - 3-PLANAR-COLOR is NP-complete

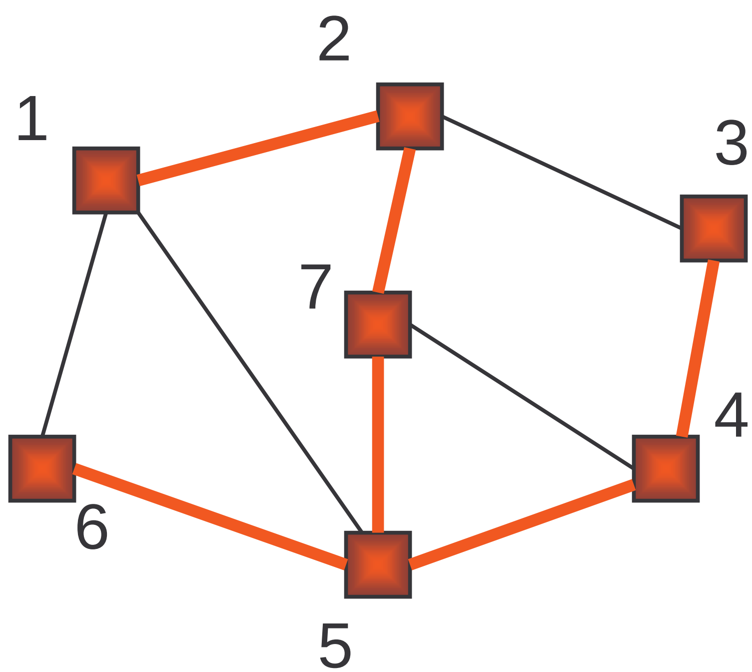
Boundary Continued

- Independent Set
  – Maximum independent set is NP-hard
  – Maximal independent set is in P
- Cutting a graph
  – Maximum cut in a graph is NP-hard
  – Minimum cut in a graph is in P (equivalent to Max Flow)
- Spanning Tree
  – Minimum spanning tree is in P
  – Degree constrained spanning tree is NP-hard
  – Bounded diameter spanning tree is NP-hard

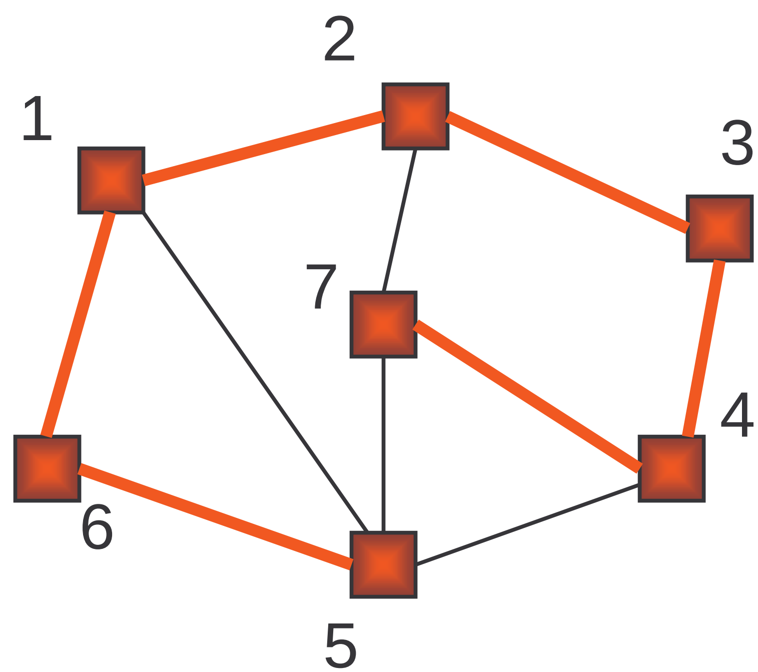
Load Balanced Spanning Tree

- Input: An undirected graph $G = (V,E)$.
- Output: A number k and a spanning tree (V,T) of degree k. Furthermore, there is no spanning tree of degree $< k$.

Spanning Tree of Degree 3



Spanning Tree of Degree 2



LBST Decision Problem

- Input: An undirected graph $G = (V,E)$ and number k.
- Output: Determine if G has a spanning tree of degree k.

Hamiltonian Path Decision Problem

- Input: Undirected Graph G =(V,E).
- Output: Determine if there is a path in G that visits each node exactly once.

- Hamiltonian Path is known to be NP-complete

Hamiltonian Path is Polynomial time Reducible to Spanning Tree of Degree 2

- If there an algorithm to quickly determine if a graph has a spanning tree of degree 2 then there is an algorithm to quickly solve the Hamiltonian path problem.
  - A spanning tree of degree 2 is a Hamiltonian path!
  - These problems are essentially the same problem.

Lessons When Coping

- Lesson 1. Any problem that is in NP may be NP-complete.
- Lesson 2. Any problem in NP may be in P.
- Lesson 3. You may not be able to determine either
  - factoring is open
  - graph isomorphism is open