CSEP 521 - Applied Algorithms

Linear Programming

Reading:

Skiena, Section 8.2.6

CLRS, Chapter29 (2^{nd} ed. Only).

"Linear Algebra and Its Applications", by Gilbert Strang, by Gilbert Strang, chapter 8

"Linear Programming", by Vasek Chvatal

"Introduction to Linear Optimization", by Dimitris Bertsimas and John Tsitsiklis

Linear Programming

- The process of minimizing a linear objective function subject to a finite number of linear equality and inequality constraints.
- The word "programming" is historical and predates computer programming.
- Example applications:
  - airline crew scheduling
  - manufacturing and production planning
  - telecommunications network design
- "Few problems studied in computer science have greater application in the real world."

An Example: The Diet Problem

- A student is trying to decide on lowest cost diet that provides sufficient amount of protein, with two choices:
  - steak: 2 units of protein/pound, $3/pound
  - peanut butter: 1 unit of protein/pound, $2/pound
- In proper diet, need 4 units protein/day.

Let x = # pounds peanut butter/day in the diet.

Let y = # pounds steak/day in the diet.

Goal: minimize $2x + 3y$ (total cost)
subject to constraints:

$x + 2y \geq 4$

$x \geq 0, \; y \geq 0$

This is an LP- formulation of our problem

An Example: The Diet Problem

Goal: minimize $2x + 3y$ (total cost)
subject to constraints:

$x + 2y \geq 4$

$x \geq 0, \; y \geq 0$

- This is an optimization problem.
- Any solution meeting the nutritional demands is called a *feasible solution*
- A feasible solution of minimum cost is called the *optimal solution*.

Linear Program - Definition

A linear program is a problem with n variables $x_1,\dots,x_n$, that has:

1. A linear objective function, which must be minimized/maximized. Looks like:

   min (max) $c_1x_1+c_2x_2+\dots+c_nx_n$

2. A set of m linear constraints. A constraint looks like:

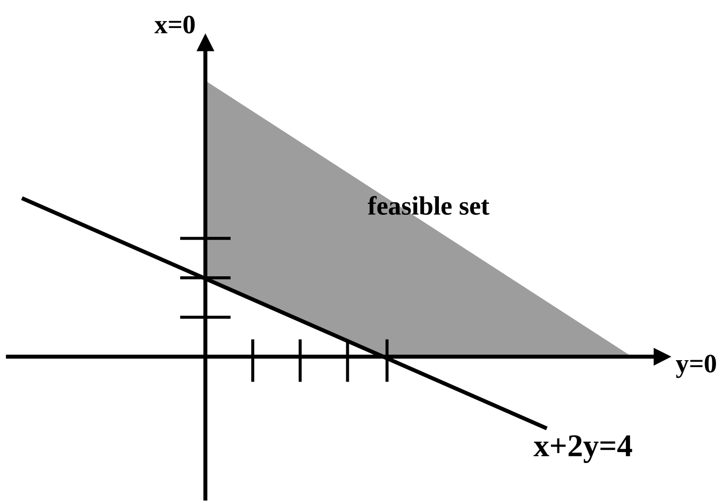
   $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i$ (or $\ge$ or $=$)

Note: the values of the coefficients c_i, $a_{i,j}$ are given in the problem input.

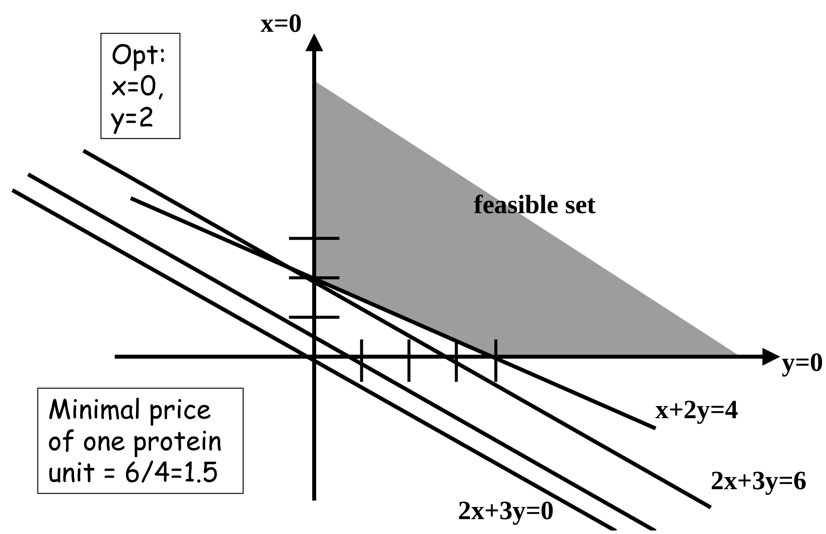
Feasible Set

- Each linear inequality divides n-dimensional space into two halfspaces, one where the inequality is satisfied, and one where it's not.
- Feasible Set : solutions to a family of linear inequalities.
- The linear cost functions, defines a family of parallel hyperplanes (lines in 2D, planes in 3D, etc.). Want to find one of minimum cost → must occur at corner of feasible set.

Visually…
x= peanut butter, y = steak



Optimal vector occurs at some corner of the feasible set!



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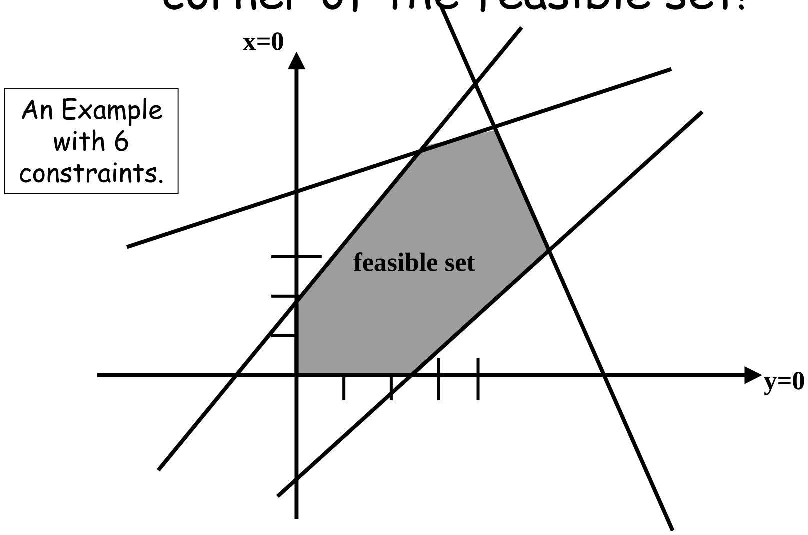


General Form of a Linear Program.

Minimize $b_1y_1 + b_2y_2 + \ldots + b_my_m$
subject to $\Sigma_{1 \le i \le m} a_{ij}y_i \ge c_j \quad j=1..n$
$y_i \ge 0 \quad i=1..m$

or

Maximize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
subject to $\Sigma_{1 \le j \le n} a_{ij}x_j \le b_j \quad i=1..m$
$x_j \ge 0 \quad j=1..n$

The Feasible Set

- Intersection of a set of half-spaces, called a polyhedron.
- If it's bounded and nonempty, it's a polytope.

There are 3 cases:

- feasible set is empty.
- cost function is unbounded on feasible set.
- cost has a minimum (or maximum) on feasible set.

First two cases very uncommon for real problems in economics and engineering.

Solving LP

- There are several polynomial-time algorithms that solve any linear program optimally.
  - The Simplex method (see bonus slides)
  - The Elipsoid method
  - More
- These algorithms can be implemented in various ways.
- There are many existing software packages for LP.
- It is convenient to use LP as a ``black box'' for solving various optimization problems.

LP formulation: another example

Bob's bakery sells bagel and muffins.

To bake a dozen of bagels Bob needs 5 cups of flour, 2 eggs, and one cup of sugar.

To bake a dozen of muffins Bob needs 4 cups of flour, 4 eggs and two cups of sugar.

Bob can sell bagels in 10$/dozen and muffins in 12$/dozen.

Bob has 50 cups of flour, 30 eggs and 20 cups of sugar.

How many bagels and muffins should Bob bake in order to maximize his revenue?

LP formulation: Bob's bakery

	Bagels	Muffins	Avail.
Flour	5	4	50
Eggs	2	4	30
Sugar	1	2	20
Revenue	10	12	

$$a = \begin{pmatrix} 5 & 4 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 10 & 12 \end{pmatrix} \quad c = \begin{pmatrix} 50 \\ 30 \\ 20 \end{pmatrix}$$

Maximize $10x_1+12x_2$

s.t. $5x_1+4x_2 \le 50$

$2x_1+4x_2 \le 30$

$x_1+2x_2 \le 20$

$x_1 \ge 0, x_2 \ge 0$

Maximize $b \cdot x$

s.t. $ax \le c$

$x \ge 0.$

In class exercise: Formulate as LP

You want to invest $1000 in 3 stocks, at most $400 per stock

	price/share	dividends/year
stock A	$50	$2
stock B	$200	$5
stock C	$20	0

Stock C has probability ½ of appreciating to $25 in a year, and prob ½ of staying $20.

What amount of each stock should be bought to maximize dividends + expected appreciation over a year?

In class exercise: Formulate as LP

Solution: Let x_a, x_b, and x_c denote the amounts of A,B,C stocks to be bought.

Objective function:

Constraints:

Example: Max Flow

Variables: f(e) - the flow on edge e.

Max $\Sigma_{e\in in(s)}$ f(e)

s.t.

$f(e) \leq c(e), \ \forall e \in E$ (Edge condition)

$\Sigma_{e \in in(v)} f(e) - \Sigma_{e \in out(v)} f(e) = 0 \ , \forall v \in V\text{-}\{s,t\}$

(Vertex condition)

$f(e) \geq 0, \ \forall e \in E$

A Central Result of LP Theory: Duality Theorem

- Every linear program has a dual
- If the original is a minimization, the dual is a maximization and vice versa
- Solution of one leads to solution of other

Primal: Minimize **cx** subject to $A\mathbf{x} \geq \mathbf{b}$, $\mathbf{x} \geq 0$

Dual: Maximize **yb** subject to $\mathbf{y}A^T \leq \mathbf{c}$, $\mathbf{y} \geq 0$

If one has optimal solution so does other, and their values are the same.

Primal: Minimize **cx** subject to $A\mathbf{x} \geq \mathbf{b}$, $\mathbf{x} \geq 0$

Dual: Maximize **yb** subject to $\mathbf{y}A^T \leq \mathbf{c}$, $\mathbf{y} \geq 0$

- In the primal, **c** is cost function and **b** was in the constraint. In the dual, reversed.
- Inequality sign is changed and minimization turns to maximization.

Primal:

minimize 2x + 3y

s.t $x+2y \geq 4$,

$2x + 5y \geq 1$,

$x - 3y \geq 2$,

$x \geq 0, y \geq 0$

Dual:

maximize 4p +q + 2r

s.t $p+2q + r \leq 2$,

$2p+5q -3r \leq 3$,

$p,q,r \geq 0$

Simple Example

- Diet problem: minimize 2x + 3y

  subject to $x+2y \geq 4$,

  $x \geq 0, y \geq 0$
- Dual problem: maximize 4p

  subject to $p \leq 2$,

  $2p \leq 3$,

  $p \geq 0$
- Dual: the problem faced by a druggist who sells synthetic protein, trying to compete with peanut butter and steak

Simple Example

- The druggist wants to maximize the price p, subject to constraints:
  - synthetic protein must not cost more than protein available in foods.
  - price must be non-negative or he won't sell any
  - revenue to druggist will be 4p
- Solution: $p \leq 3/2$ → objective value = 4p = 6
- Not coincidence that it's equal the minimal cost in original problem.

What's going on?

- Notice: feasible sets completely different for primal and dual, but nonetheless an important relation between them.
- Duality theorem says that in the competition between the grocer and the druggist the result is always a tie.
- Optimal solution to primal tells purchaser what to do.
- Optimal solution to dual fixes the natural prices at which economy should run.
- The diet x and vitamin prices y are optimal when
  - grocer sells zero of any food that is priced above its vitamin equivalent.
  - druggist charges 0 for any vitamin that is oversupplied in the diet.

Duality Theorem

Druggist's max revenue = Purchasers min cost

Practical Use of Duality:

- Sometimes simplex algorithm (or other algorithms) will run faster on the dual than on the primal.
- Can be used to bound how far you are from optimal solution.
- Important implications for economists.

Linear Programming, Mid-Summary

- Of great practical importance to solve linear programs:
  - they model important practical problems
    - production, approximating the solution of inconsistent equations, manufacturing, network design, flow control, resource allocation.
  - solving an LP is often an important component of solving or approximating the solution to an **integer linear programming problem**.
- Can be solved in poly-time, the simplex algorithm works very well in practice.
- One problem where you really do not want to roll your own code.

LP-based approximations

- We don't know any polynomial-time algorithm for any NP-complete problem
- We know how to solve LP in polynomial time
- We will see that LP can be used to get approximate solutions to some NP-complete problems.

Weighted Vertex Cover

Input: Graph G=(V,E) with non-negative weights w(v) on the vertices.

Goal: Find a minimum-cost set of vertices S, such that all the edges are covered. An edge is covered iff at least one of its endpoints is in S.

Recall: Vertex Cover is NP-complete.

The best known approximation factor is

2- (log log |V|/2 log|V|).

Weighted Vertex Cover

Variables: for each $v \in V$, x(v) – is v in the cover?

Min $\Sigma_{v\in V}$ w(v)x(v)

s.t.

$x(v) + x(u) \geq 1, \ \forall (u,v) \in E$

$x(v) \in \{0,1\} \quad \forall v \in V$

The LP Relaxation

This is **not** a linear program: the constraints of type $x(v) \in \{0,1\}$ are not linear.

Such problems (LP's with integrality constraints on variables) are called **integer linear programs (IP).** Solving IP's is an NP-hard problem.

However, if we replace the constraints $x(v) \in \{0,1\}$ by $x(v) \geq 0$ and $x(v) \leq 1$, we will get a linear program.

The resulting LP is called a **Linear Relaxation** of IP, since we relax the integrality constraints.

LP Relaxation of Weighted Vertex Cover

Min $\Sigma_{v \in V} w(v)x(v)$

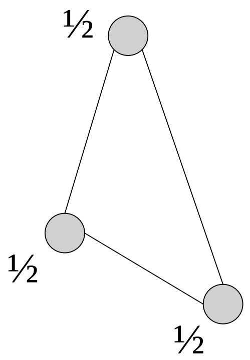
s.t.

$x(v) + x(u) \geq 1, \ \forall (u,v) \in E$

$x(v) \geq 0, \ \forall v \in V$

$x(v) \leq 1, \ \forall v \in V$

LP Relaxation of Weighted Vertex Cover - example



Consider the case in which all weights are 1.

An optimal VC has cost 2 (any two vertices)

An optimal relaxation has cost 3/2 (for all three vertices x(v)=1/2)

The LP and the IP are different problems. Can we still learn something about Integer VC?

Why LP Relaxation Is Useful ?

The optimal value of LP-solution provides a bound on the optimal value of the original optimization problem. OPT(LP) is always better than OPT(IP) (why?)

Therefore, if we find an integral solution within a factor r of OPT_{LP}, it is also an r-approximation of the original problem.

These can be done by 'wise' rounding.

Approximation of Vertex Cover Using LP-Rounding

1. Solve the LP-Relaxation.

2. Let S be the set of all the vertices v with $x(v) \geq 1/2$. Output S as the solution.

Analysis: The solution is feasible: for each edge e=(u,v), either $x(v) \geq 1/2$ or $x(u) \geq 1/2$

The value of the solution is: $\Sigma_{v \in s} w(v) = \Sigma_{\{v|x(v) \geq 1/2\}} w(v) \leq 2\Sigma_{v \in V} w(v)x(v) = 2OPT_{LP}$

Since $OPT_{LP} \leq OPT_{VC}$, the cost of the solution is $\leq 2OPT_{VC}$.

Bonus material: The Simplex Method

- Phase I : locate a corner of the feasible set.
  - corner = intersection of n different planes (in n dimensions)
- Phase II: move from corner to corner along the edges of the feasible set -- always go along an edge that is guaranteed to decrease the cost.
  - Edge = intersection of n-1 different planes
- When reach a local minimum (maximum), you've found the optimum.

Simplex Algorithm: An Example in 3D

Maximize $5x + 4y + 3z$

subject to $2x + 3y + z \leq 5$

$4x + y + 2z \leq 11$

$3x + 4y + 2z \leq 8$

$x,y,z \geq 0$.

Step 0: convert inequalities into equalities by introducing slack variables a,b,c.

Define: $a = 5-2x-3y-z$ → $a \geq 0$

$b = 11-4x-y-2z$ → $b \geq 0$

$c = 8-3x-4y-2z$ → $c \geq 0$

$F = 5x+4y+3z$, objective function

Example of Simplex Method, continued.

Step 1: Find initial feasible solution:

$x=0, y=0, z=0$ ➔ $a=5, b=11, c=8$ ➔ $F=0$.

Step 2: Find feasible solution with higher value of F

For example, can increase x to get $F=5x$.

How much can we increase x?

$a = 5-2x-3y-z \geq 0$ ➔ $x \leq 5/2$ **most stringent**

$b = 11-4x-y-2z \geq 0$ ➔ $x \leq 11/4$

$c = 8-3x-4y-2z \geq 0$ ➔ $x \leq 8/3$

➔ increase x to 5/2 ➔ **$F= 25/2$**, $a=0, b=1, c=1/2$

Example of Simplex Method, continued.

Want to keep doing this, need to get back into state where x,b,c on l.h.s. of equations.

$a = 5-2x-3y-z$ → $x= 5/2 -3/2\, y -1/2\, z - 1/2\, a$ (*)

Substituting (*) into other equations:

$b = 11-4x-y-2z \geq 0$ ➔ $b = 1+5y+2a$

$c = 8-3x-4y-2z \geq 0$ ➔ $c = 1/2+1/2\, y -1/2\, z + 3/2\, a$

$F = 5x+4y + 3z$ ➔ $F= 25/2-7/2\, y +1/2\, z -5/2\, a$

In order to increase F again, should increase z

Example of Simplex Method, continued.

How much can we increase z?

$x = 5/2 - 3/2\, y - 1/2\, z - 1/2\, a$ → $z \leq 5$

$b = 1 + 5y + 2a$ → no restriction

$c = 1/2 + 1/2\, y - 1/2\, z + 3/2\, a$ → $z \leq 1$ **most stringent (^)**

Setting $z = 1$ yields
$x=2, y=0, z=1, a=0, b = 1, c = 0$.

$F= 25/2 - 7/2\, y + 1/2\, z - 5/2\, a$ → $F = 13$.

Again, construct system of equations.
From (^) $z = 1 + y + 3a - 2c$.

Example of Simplex Method, continued.

Substituting back into other equations:

$z = 1 + y + 3a - 2c$.

$x= 5/2 - 3/2\, y - 1/2\, z - 1/2\, a$ → $x = 2-2y-2a + c$

$b = 1 + 5y + 2a$ → $b= 1 + 5y + 2a$

$F = 25/2 - 7/2\, y + 1/2\, z - 5/2\, a$ → $F = 13 - 3y - a - c$

And we're done.