CSEP 521 – Applied Algorithms

Graph Algorithms

Broadcasting in a Network

DFS, BFS,

Shortest Path Problems

Graph Algorithms - reading

DFS, BFS -

CLRS: chapter 22 (1st and 2nd editions.)

Skiena: section 4.4

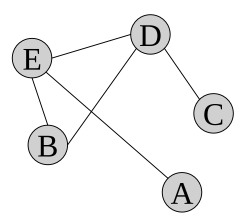
Shortest Path Problems –

CLRS: chapter 25 (1st ed.) or 24 (2nd ed.)

Skiena: section 4.8

Definition

- A graph G is given by the two sets V and E.
- V is a set of points (vertices)
- E is a set of lines (edges) connecting pairs of points.



- Examples:
  - –airline flight map.
  - –communication networks.
  - –precedence constraints on the scheduling of jobs.
  - –flow networks.

V={A,B,C,D,E}

E={(B,E),(E,D),(D,C),(B,D),(A,E)}

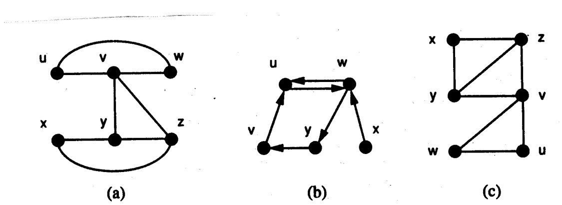


Figure 12.1 Three graphs. (a) and (c) are undirected graphs, (b) is a directed graph. The placement of the vertices on the paper is immaterial when we draw graphs; for example, (a) and (c) are in fact depictions of the same graph.

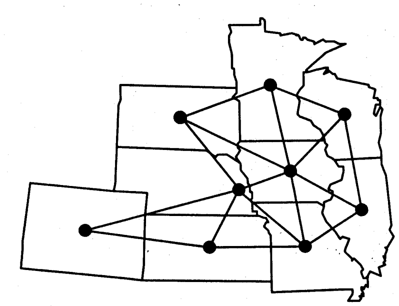


Figure 12.2 A map and its associated undirected graph. Each region is represented by a vertex, and an edge joins each pair of vertices that correspond to bordering regions.

A More Detailed Definition

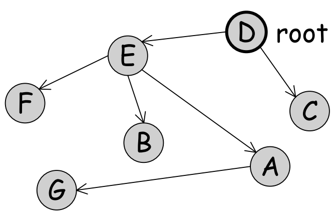
- An **undirected graph** is a pair (V,E), where V is a finite set, and E is a set of *unordered* pairs (u,v), where u and v are in V.
- Terminology: If (u,v) is an edge (i.e., in E):
  u and v are **adjacent**; v is a **neighbor** of u.
- A **directed** graph is a pair (V,E), where V is a finite set, and E is a set of *ordered* pairs (u,v) (both in V).

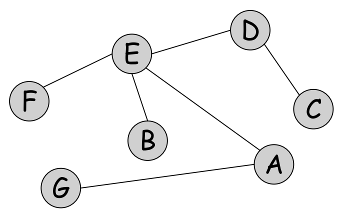
More Definitions

- Let n = |V|, m = |E|
- The **size** of the graph is n+m
  - Any algorithm that needs to inspect each vertex and edge has running time $\Omega(n+m)$
- A **path** in G is a sequence $(v_0,v_1,\ldots,v_k)$ of vertices such that $(v_i, v_{i+1}) \in E$, for all $0 \le i < k$. Its **length** is k and it is a path from v_0 to v_k.
- A **cycle** is a path such that $v_0 = v_k$.
- An undirected graph is **connected** if and only if there is a path between every pair of vertices.
- In any connected graph m= $\Omega(n)$.

Trees

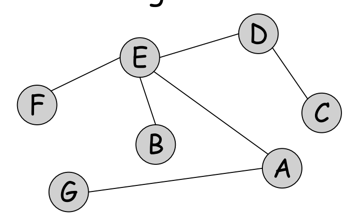
- An undirected graph is a tree if it is connected and contains no cycles.
- A directed graph is a directed tree if it has a root and its underlying undirected graph is a tree.
- $r\in V$ is a root if every vertex $v\in V$ is reachable from r; i.e., there is a directed path which starts in r and ends in v.





Alternative Definitions of Undirected Trees

- G is cycles-free, but if any new edge is added to G, a cycle is formed.
- for every pair of vertices u,v, there is a unique, simple path from u to v.
- G is connected, but if any edge is deleted from G, the connectivity of G is interrupted.
- G is connected and has n–1 edges.



G is a tree $\Rightarrow$ G is cycle-free and has $n-1$ edges.

$\Rightarrow$ We show, by induction on n, that if G is a tree (cycle-free and connected), then its number of edges is n–1.

Base: n=1 

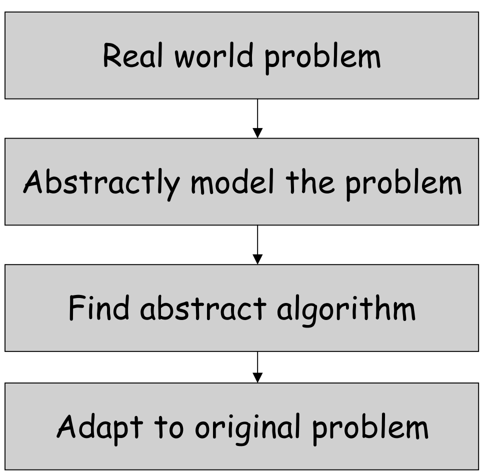
Step: Assume that it is true for all n < m, and let G be a tree with m vertices. Delete from G any edge e. By definition (3), G is not connected any more, and is broken into two connected components each of which is cycle-free and therefore is a tree. By the inductive hypothesis, each component has one edge less than the number of vertices. Thus, both have m–2 edges. Add back e, to get m–1.

More Definitions

- A **subgraph** of a graph $G=(V,E)$ is a graph $G'=(V',E')$ such that $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$.
- A **connected component** of an undirected graph G is a maximal connected subgraph of G.

Enough with the definitions. Let's do something.

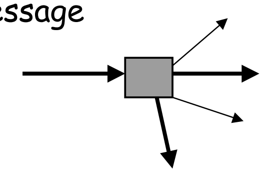
Applied Algorithm Scenario



Broadcasting in a Network

- Network of Routers
  - Organize the routers to efficiently broadcast messages to each other.

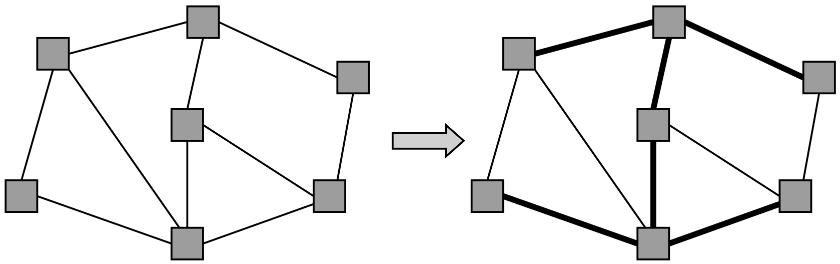
Incoming message



- Duplicate and send to some neighbors.
- Eventually all routers get the message

Goal: Minimize the number of messages.

Spanning Tree in a Graph



Vertex = router
Edge = link between routers

Spanning tree
- Connects all the vertices
- No cycles

Spanning Tree Problem

- Input: An undirected graph G = (V,E). G is connected.
- Output: T contained in E such that
  - (V,T) is a connected graph
  - (V,T) has no cycles

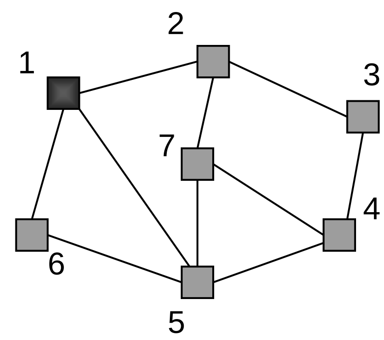
Depth First Search Algorithm

- Recursive marking algorithm
- Initially every vertex is unmarked

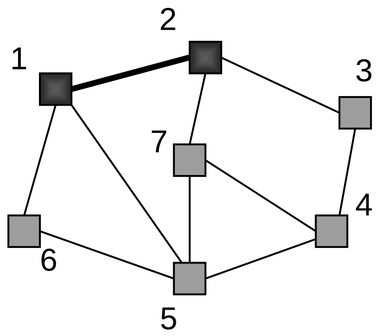
```
DFS(i: vertex)
  mark i;
  for each j adjacent to i do
    if j is unmarked then DFS(j)
end{DFS}
```


Example of Depth First Search

DFS(1)

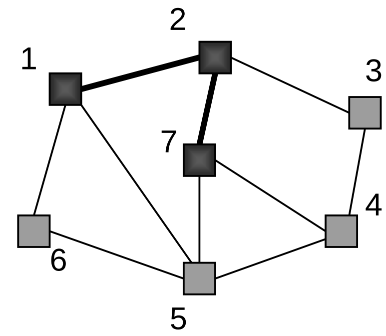


Example Step 2



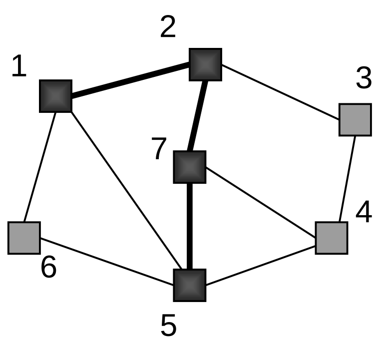
DFS(1)
DFS(2)

Example Step 3



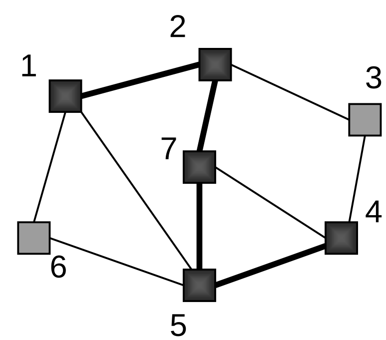
DFS(1)
DFS(2)
DFS(7)

Example Step 4



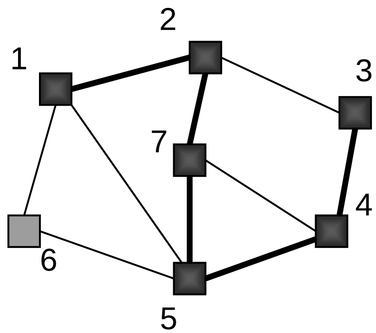
DFS(1)
DFS(2)
DFS(7)
DFS(5)

Example Step 5



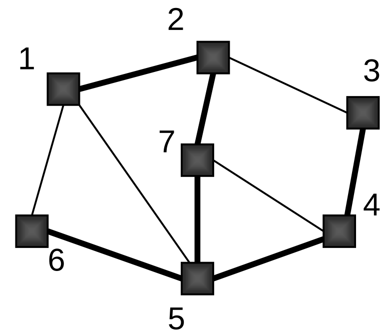
DFS(1)
DFS(2)
DFS(7)
DFS(5)
DFS(4)

Example Step 6



DFS(1)
DFS(2)
DFS(7)
DFS(5)
DFS(4)
DFS(3)

Example Step 7



DFS(1)
DFS(2)
DFS(7)
DFS(5)
DFS(4)
DFS(3)
DFS(6)

Note that the edges traversed in the depth first search form a spanning tree.

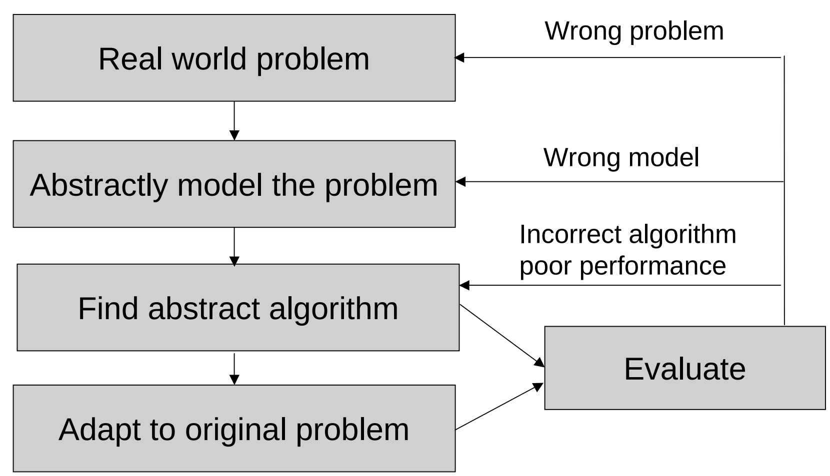
Spanning Tree Algorithm

```
Main
T := empty set;
ST(1);
end{Main}
```

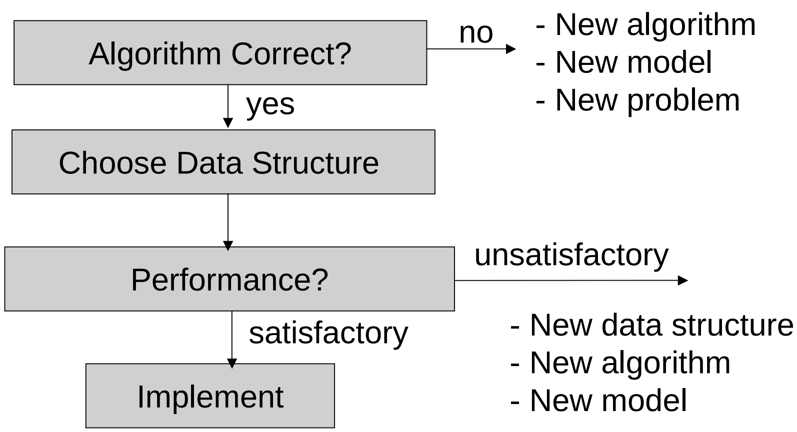
```
ST(i: vertex)
   mark i;
   for each j adjacent to i do
      if j is unmarked then
         Add {i,j} to T;
         ST(j);
end{ST}
```

The addition to DFS

Applied Algorithm Scenario

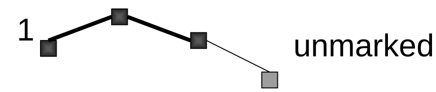


Evaluation Step Expanded



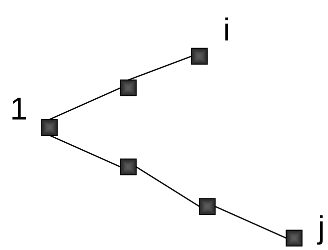
Correctness of ST Algorithm

- There are no cycles in T
  - This is an invariant of the algorithm.
  - Each edge added to T goes from a vertex in T to a vertex not in T.
- If G is connected then eventually every vertex is marked.

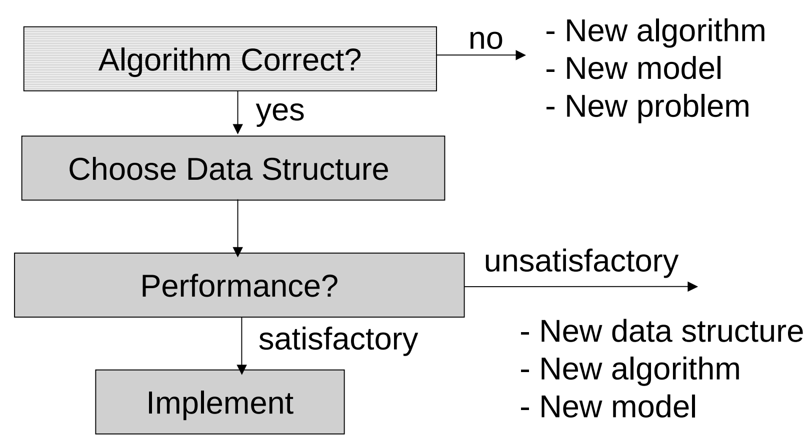


Correctness (cont.)

- If G is connected then so is (V,T)



Data Structure Step

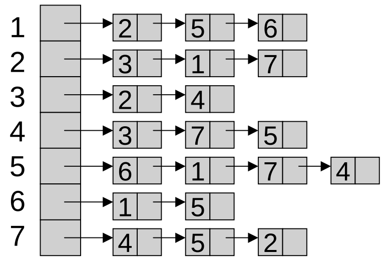


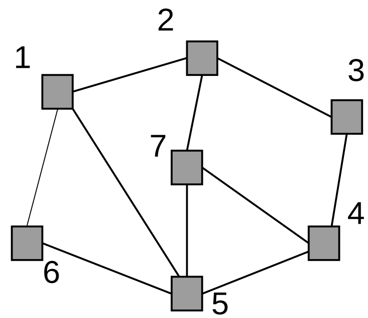
Edge List and Adjacency Lists

- List of edges

1	5	1	2	2	3	5	7	5	5
2	1	6	7	3	4	6	4	7	4

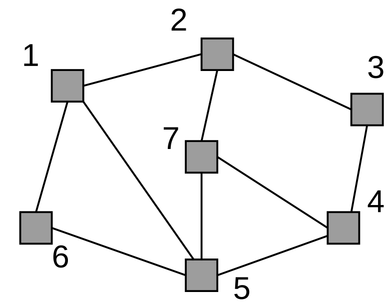
- Adjacency lists





Adjacency Matrix

	1	2	3	4	5	6	7
1	0	1	0	0	1	1	0
2	1	0	1	0	0	0	1
3	0	1	0	1	0	0	0
4	0	0	1	0	1	0	1
5	1	0	0	1	0	1	1
6	1	0	0	0	1	0	0
7	0	1	0	1	1	0	0



Data Structure Choice

- Edge list
  - Simple but does not support depth first search
- Adjacency lists
  - Good for sparse graphs
  - Supports depth first search
- Adjacency matrix
  - Good for dense graphs
  - Supports depth first search

Spanning Tree with Adjacency Lists

```
Main
   G is array of adjacency lists;
   M[i] := 0 for all i;
   T is empty;
   Spanning_Tree(1);
end{Main}
```

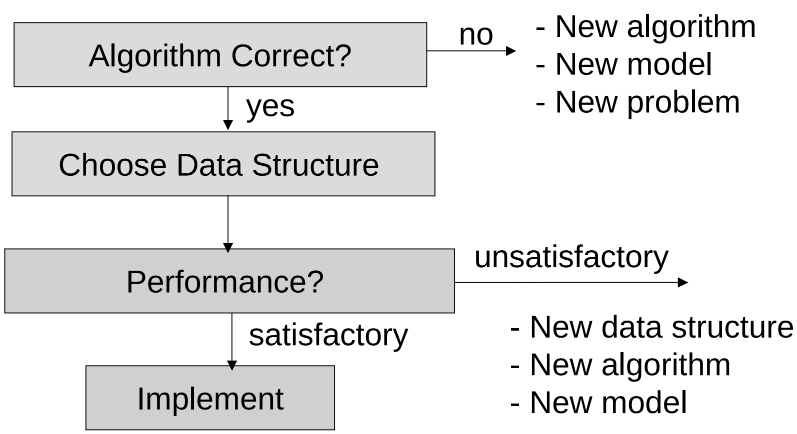
M is the marking array.

Node of linked list:

vertex [|] next

```
ST(i: vertex)
   M[i] := 1;
   v := G[i];
   while (v ≠ null)
      j := v.vertex;
      if (M[j] = 0) then
         add {i,j} to T;
         ST(j);
      v := v.next;
end{ST}
```

Performance Step



Performance of ST Algorithm

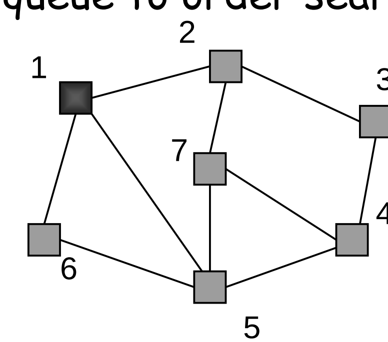
- n vertices and m edges
- Connected graph ($m \geq n-1$)
- Storage complexity O(m)
- Time complexity O(m) - for each edge we perform O(1) operations in each of the two endpoints.

Other Uses of Depth First Search

- Popularized by Hopcroft and Tarjan 1973
- Connected components
- Strongly connected components in directed graphs
- Topological sorting of a acyclic directed graphs
- Maze solving

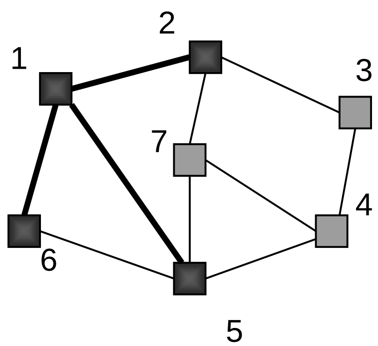
ST using Breadth First Search 1

- Uses a queue to order search



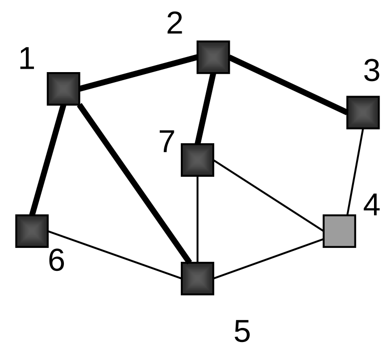
Queue = 1

Breadth First Search 2



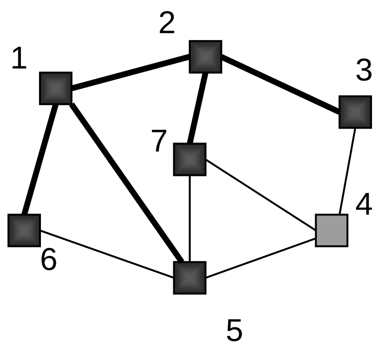
Queue = 2,6,5

Breadth First Search 3



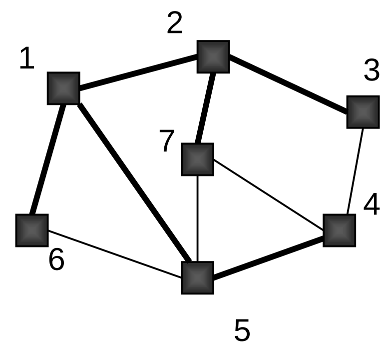
Queue = 6,5,7,3

Breadth First Search 4



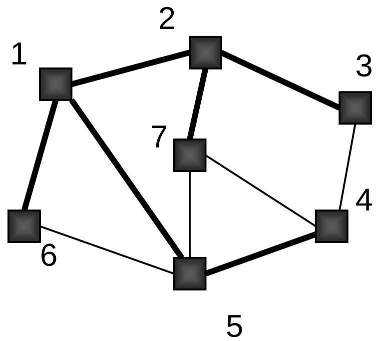
Queue = 5,7,3

Breadth First Search 5



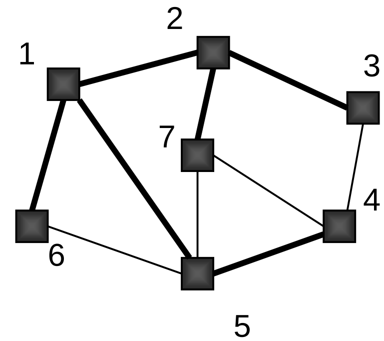
Queue = 7,3,4

Breadth First Search 6



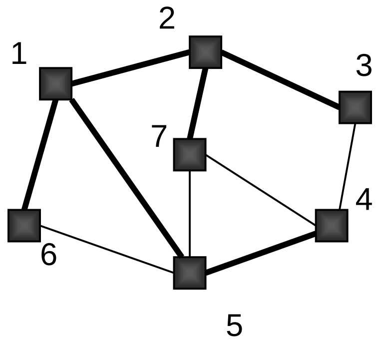
Queue = 3,4

Breadth First Search 7



Queue = 4

Breadth First Search 8



Queue =

Spanning Tree using Breadth First Search (BFS)

```
Initialize T to be empty;
Initialize Q to be empty;
Enqueue(1,Q) and mark 1;
while (Q is not empty) do
    i := Dequeue(Q);
    for each j adjacent to i do
        if j is not marked then
            add {i,j} to T;
            mark j;
            Enqueue(j,Q);
```

Depth First vs Breadth First

- Depth First
  - Stack or recursion
  - Many applications
- Breadth First
  - Queue (recursion no help)
  - Can be used to find shortest paths from the start vertex
- Both are O(|E|)

Shortest-path Algorithms

- Scenario: One router creates messages (source). Each message needs to reach other routers (one or more) along the shortest possible path.
- Abstraction: given a vertex s, find the shortest path from s to any other vertex of G.
- Other shortest path problems:
  - Different edges have different lengths (delay, cost, etc.)
  - All-pair shortest path problem: no specific source.

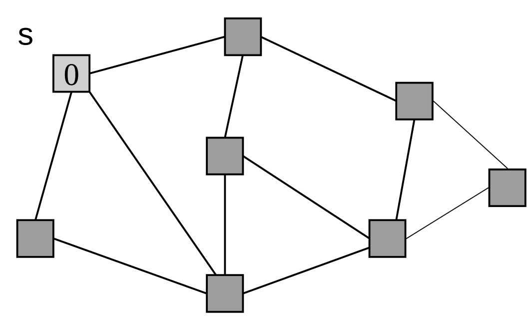
Using BFS for Shortest-path

- Given a vertex s, find the shortest path from s to any other vertex of G.

<u>A 'centralized' version of BFS:</u>

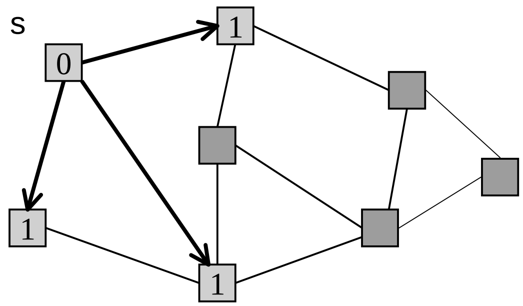
1. Label vertex s with 0.
2. $i \leftarrow 0$
3. Find all unlabeled vertices adjacent to at least one vertex labeled i. If none are found, stop.
4. Label all the vertices found in (3) with i + 1.
5. $i \leftarrow i + 1$ and go to (3).

BFS for Shortest Path (i=0)



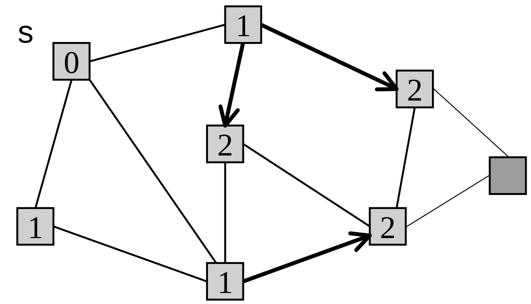
Vertices whose distance from s is 0 are labeled.

BFS for Shortest Path (i=1)



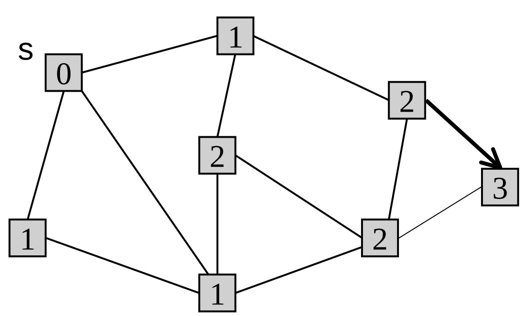
Vertices whose distance from s is 1 are labeled.

BFS for Shortest Path (i=2)



Vertices whose distance from s is 2 are labeled.

BFS for Shortest Path (i=3)



Vertices whose distance from s is 3 are labeled.

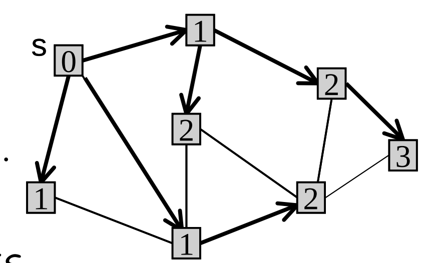
In the next iteration we find out that the whole graph is labeled and we stop.

The BFS Tree

Theorem: Each vertex is labeled by it its length from s.

Proof: By induction on the label.

For any v≠s, let p(v) be the vertex that 'discovered' v in BFS.



Then T={(p(v),v)} is a directed spanning tree rooted in s, and for each vertex v, the path from s to v in T is a shortest path from s to v in G.

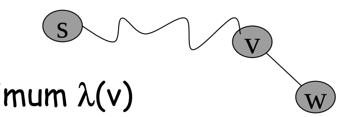
Note: the 'centralized' version is for simplification only. When implemented, we need the queue as before.

Single-Source Shortest Paths (Dijkstra's algorithm)

- Using BFS, we solve the problem of finding shortest path from s to any vertex v.
- What if edges have associated costs or distances? (BFS assumes edge costs are all 1.)
- Assume each edge (u,v) has non-negative weight c(u,v).
- A weight of a path = total weights of all edges on path.
- Problem: Find, for each vertex v, a shortest (minimum weight) path from s to v.

Idea of Dijkstra's Algorithm:

- Maintain:
  - $\lambda[0..n-1]$ where $\lambda(v)$ is the cost of best path from s to v found so far, and
  - T, set of vertices v for which $\lambda(v)$ is not yet known to be optimal.
- Initially:
  - $\lambda(s) = 0$; $\lambda(v) = \infty$ for all v other than s.
  - T = V.
- In each step:
  - remove that v in T with minimum $\lambda(v)$
  - update those w in T s.t. (v,w) in E and $\lambda(w) > \lambda(v) + c(v,w)$.



Dijkstra's Algorithm

Assumption: $c(u,v) = \infty$ if (u,v) not in E.

1. $\lambda(s) \leftarrow 0$ **and for all** $v \neq s$,
$\lambda(v) \leftarrow \infty$.

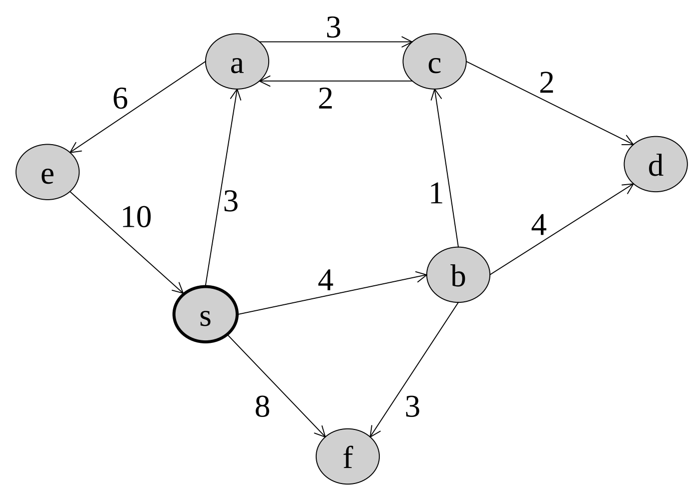
2. $T \leftarrow V$.

3. Let ***u*** **be a vertex in** ***T*** **for which** $\lambda(u)$ **is minimum.**

4. For every edge, if $v \in T$ **and**
$\lambda(v) \geq \lambda(u) + c(v,u)$ **then**
$\lambda(v) \leftarrow \lambda(u) + c(v,u)$.

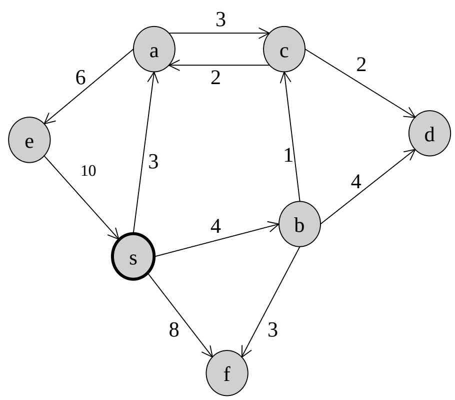
5. ***T*** **-** ***T*** **- {*u*}, if T is not empty go to step 3.**

Dijkstra's Algorithm - Example



Dijkstra's Algorithm - Example

	init	u=s	u=a
s	0	0 *	0 *
a	∞	3	3 *
b	∞	4	4
c	∞	∞	6
d	∞	∞	∞
e	∞	∞	9
f	∞	8	8

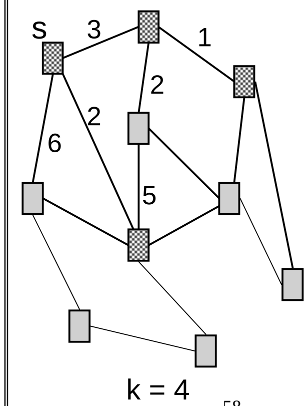


In class exercise: complete the execution.

* non-T vertices.

Why is this Algorithm Correct?

- **Theorem:** At the termination of the algorithm, λ(v) is the length of the shortest path from s to v for each vertex v of G.
- **Proof:** by induction on |V-T|.
- **Inductive hypothesis:** Let |V-T|=k.
  - ∀v in V-T, λ(v) is the length of the shortest path from s to v.
  - the vertices in V-T are the k closest vertices to s.
  - ∀v in T, λ(v) is the length of the shortest path from s to v that only goes through vertices in V-T.



Why is this Algorithm Correct?

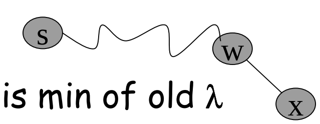
- **Base case: |V-T|=1, T=V-{s}.**
  - for every v in V-T, λ(v) is the length of shortest path from s to v.
  - ✓ we init λ(s) =0.
  - the vertices in V-T are the k closest vertices to s.
  - ✓ V-T={s}. s is surely the closest to s.
  - for every v in T, λ(v) is the length of shortest path from s to v that only goes through vertices in V-T.
  - ✓ At this stage, λ(v) = ∞ for all v in V-T.

The λ values of vertices in V-T are correct and for each such v, the shortest path from s to v only goes through vertices in V-T

- Induction Step: Suppose true for first k steps. The SP to the (k+1)[st] closest vertex, say w, can go through only vertices in V-T, otherwise, there would be a closer vertex. Therefore, when selecting the min, we select the (k+1)[st] closest vertex to s.

  Say w is added.

  New λ value for a vertex x is min of old λ value and λ(w) + c(w,x)



Dijktra's Algorithm - Run Time Analysis

Implementation 1:
- Adjacency lists.
- An array for the λ values.

Complexity:

In each iteration:

1. Finding a vertex u in T with minimal λ

   In the whole execution: $n+(n-1)+(n-2)+…+1 = O(n^2)$

2. Updating the λ-values of u's neighbors:

   In each iteration we check degree(u) values.

   The total sum of the degrees in 2m → O(m)

All together: $O(m+n^2)= O(n^2)$ (remember, $m \leq n(n-1)$)

Dijktra's Algorithm - Run Time Analysis

- Implementation 2: data structure: priority queue
- Stores set S (in our case, this is T) such that there is a linear order on key values (in our case the key is the λ value).
- Supports operations:
  - Insert(x) - insert element with key value x into set.
  - FindMin() - return value of smallest element in set.
  - DeleteMin() - delete smallest element in set.
- and usually:
  - Lookup(x), Delete(x)

Priority-Queue Implementations

- Priority-Queue can be implemented such that each of these operations takes O(log n) time for sets of size n.

Running time of Dijkstra's algorithm:

We need to consider insertions, delete Mins, lookups, modifying λ values.

Running Time of Dijkstra's Algorithm:

- n insertions: O(n log n) time
- n deleteMins: O(n log n) time
- m lookups: O(m log n) time
- m λ value mods: O(m log n) time

- Running time: O((n + m) log n))

- The $O(n^2)$ is better for dense graphs

Single-Source Shortest Paths (Bellman-Ford's algorithm)

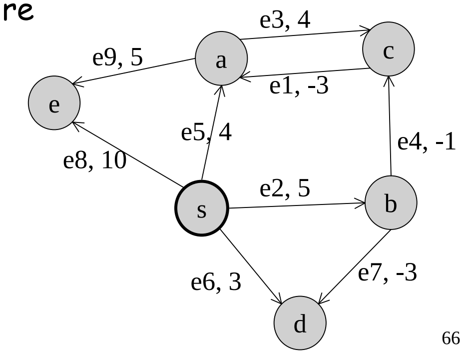
- each edge (u,v) has a weight c(u,v).
- c(u,v) might be negative, but there are no negative cycles.

1. $\lambda(s) \leftarrow 0$ and for every $v \neq s$, $\lambda(v) \leftarrow \infty$.
2. As long as there is an edge such that $\lambda(v) > \lambda(u) + c(e)$ replace $\lambda(v)$ by $\lambda(u) + c(e)$.

For our purposes ∞ is not greater than $\infty + k$, even if k is negative.

Bellman-Ford algorithm

- How do we implement this algorithm?
- Order the edges: $e_1, e_2, \ldots, e_{|E|}$.
- Perform step 2 by first checking e_1, then e_2, etc., After the first such sweep, go through additional sweeps, until an entire sweep produces no improvement.
- Running Example:



BF algorithm – correctness and run time analysis

- Theorem: if a shortest path from s to v consists of k edges, then by the end of the k^{th} sweep v will have its final label.
- Proof: induction on k (not here).
- Since k is bounded by |V| (remember, no negative cycles), step 2 is performed at most $|E| \cdot |V|$ times.
- Each comparison in step 2 can takes O(1) if the graph is kept in an Adjacency Matrix (with the weights) and an array with the $\lambda(v)$ values.

→ The time complexity of BF is $O(|E| \cdot |V|)$.