CSEP505: Programming Languages
Lecture 4: Untyped Lambda-Calculus,
Formal Operational Semantics,
…

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Where are we

- To talk about functions more precisely, we need to define them as carefully as we did IMP's constructs
- First try adding functions & local variables to IMP "on the cheap"
  - It didn't work [see last week]
- Now back up and define a language with *nothing* but functions
  - [started last week]
  - And then *encode* everything else

Review

- Cannot properly model local scope via a global heap of integers
  - Functions are not syntactic sugar for assignments to globals
- So let's build a model of this key concept
  - Or just borrow one from 1930s logic
- And for now, drop mutation, conditionals, and loops
  - We won't need them!
- The Lambda calculus in BNF

Expressions: $e ::= x \mid \lambda x.\ e \mid e\ e$
Values: $v ::= \lambda x.\ e$

That's all of it! [More review]

Expressions: $e ::= x \mid \lambda x.\ e \mid e\ e$
Values: $v ::= \lambda x.\ e$

A program is an *e*. To call a function:
substitute the argument for the bound variable
That's the key operation we were missing

Example substitutions:

$(\lambda x.\ x)\ (\lambda y.\ y) \rightarrow \lambda y.\ y$

$(\lambda x.\ \lambda y.\ y\ x)\ (\lambda z.\ z) \rightarrow \lambda y.\ y\ (\lambda z.\ z)$

$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \rightarrow (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$

Why substitution [More review]

- After substitution, the bound variable is *gone*
  - So clearly its name didn't matter
  - That was our problem before
- Given substitution we can define a little programming language
  - (correct & precise definition is subtle; we'll come back to it)
  - This microscopic PL turns out to be Turing-complete

Full large-step interpreter

```
type exp = Var of string
         | Lam of string*exp
         | Apply of exp * exp
exception BadExp
let subst e1_with e2_for x = …(*to be discussed*)
let rec interp_large e =
  match e with
   Var _ -> raise BadExp(* unbound variable *)
  | Lam _ -> e (* functions are values *)
  | Apply(e1,e2) ->
    let v1 = interp_large e1 in
    let v2 = interp_large e2 in
    match v1 with
      Lam(x,e3) -> interp_large (subst e3 v2 x)
    | _ -> failwith "impossible" (* why? *)
```

Interpreter summarized

- Evaluation produces a value `Lam(x,e3)` if it terminates
- Evaluate application (call) by
  1. Evaluate left
  2. Evaluate right
  3. Substitute result of (2) in body of result of (1)
  4. Evaluate result of (3)

A different semantics has a different *evaluation strategy*:
1. Evaluate left
2. Substitute right in body of result of (1)
3. Evaluate result of (2)

Another interpreter

```
type exp = Var of string
         | Lam of string*exp
         | Apply of exp * exp
exception BadExp
let subst e1_with e2_for x = …(*to be discussed*)
let rec interp_large2 e =
  match e with
   Var _ -> raise BadExp(*unbound variable*)
  | Lam _ -> e (*functions are values*)
  | Apply(e1,e2) ->
     let v1 = interp_large2 e1 in
     (* we used to evaluate e2 to v2 here *)
     match v1 with
       Lam(x,e3) -> interp_large2 (subst e3 e2 x)
     | _ -> failwith "impossible" (* why? *)
```


What have we done

- Syntax and two large-step semantics for the *untyped lambda calculus*
  - First was "call by value"
  - Second was "call by name"
- Real implementations don't use substitution
  - They do something *equivalent*
- Amazing (?) fact:
  - If call-by-value terminates, then call-by-name terminates
  - (They might both not terminate)

What will we do

- Go back to math metalanguage
  - Notes on concrete syntax (relates to OCaml)
  - Define semantics with inference rules
- Lambda encodings (show our language is mighty)
- Define substitution precisely
- Environments

Next time??
- Small-step
- Play with *continuations* ("very fancy" language feature)

Syntax notes

- When in doubt, put in parentheses
- Math (and OCaml) resolve ambiguities as follows:

1. λx. e1 e2 is (λx. e1 e2)
   – *not* (λx. e1) e2

General rule: Function body "starts at the dot" and "ends at the first *unmatched* right paren"

Example:
(λx. y (λz. z) w) q

Syntax notes

2. e1 e2 e3 is (e1 e2) e3
   – *not* e1 (e2 e3)

General rule: Application "associates to the left"

So e1 e2 e3 e4 is (((e1 e2) e3) e4)

It's just syntax

- As in IMP, we really care about abstract syntax
  - Here, internal tree nodes labeled "λ" or "apply" (i.e., "call")
- Previous 2 rules just reduce parens when writing trees as strings
- Rules may seem strange, but they're the most convenient
  - Based on 70 years experience
  - Especially with currying

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Inference rules

- A metalanguage for operational semantics
  - Plus: more concise (& readable?) than OCaml
  - Plus: useful for reading research papers
  - Plus: natural support for *nondeterminism*
    - *Definition* allowing observably different *implementations*
  - Minus: less tool support than OCaml (no compiler)
  - Minus: one more thing to learn
  - Minus: painful in Powerpoint

Informal idea

Want to know:
what values (0, 1, many?) an expression can evaluate to

So define a *relation* over *pairs* (**e**,**v**):

- Where **e** is an expression and **v** is a value
- Just a subset of all pairs of expressions and values

If the language is deterministic, this *relation* turns out to be a *function* from expressions to values

Metalanguage supports defining relations

- Then prove a relation is a function (if it is)

Making up metasyntax

Rather than write (**e**,**v**), we'll write *e* ⇓ *v*.

- It's just *metasyntax (!)*
  - Could use interp(e,v) or « v ☯ e » if you prefer
- Our metasyntax follows PL convention
  - Colors are not conventional (slides: green = metasyntax)
- And distinguish it from other relations

First step: define the *form* (arity and metasyntax) of your relation(s):

$\boxed{e \Downarrow v}$

This is called a *judgment*

What we need to define

So we can write *e* ⇓ *v* for any *e* and *v*

- But we want such a thing to be "true" to mean *e* can evaluate to *v* and "false" to mean it cannot

Examples (before the definition):

- (λx. λy. y x) ((λz. z) (λz. z)) ⇓ λy. y (λz. z) in the relation
- (λx. λy. y x) ((λz. z) (λz. z)) ⇓ λz. z not in the relation
- λy. y ⇓ λy. y in the relation
- (λy. y) (λx. λy. y x) ⇓ λy. y not in the relation
- (λx. x x) (λx. x x) ⇓ λy. y not in the relation
- (λx. x x) (λx. x x) ⇓ (λx. x x) (λx. x x) metasyntactically bogus

Inference rules

$e \Downarrow v$ $\quad$ $e\{v/x\} = e'$

$$\frac{}{\lambda x.\ e \Downarrow \lambda x.\ e}\ [\text{lam}]$$

$$\frac{e1 \Downarrow \lambda x.\ e3 \quad e2 \Downarrow v2 \quad e3\{v2/x\} = e4 \quad e4 \Downarrow v}{e1\ e2 \Downarrow v}\ [\text{app}]$$

- Using definition of a set of 4-tuples for substitution
  - (exp * value * variable * exp)
  - Will define substitution later

Inference rules

$e \Downarrow v$ $\quad$ $e\{v/x\} = e'$

$$\frac{}{\lambda x.\ e \Downarrow \lambda x.\ e}\ [\text{lam}]$$

$$\frac{e1 \Downarrow \lambda x.\ e3 \quad e2 \Downarrow v2 \quad e3\{v2/x\} = e4 \quad e4 \Downarrow v}{e1\ e2 \Downarrow v}\ [\text{app}]$$

- Rule top: *hypotheses* (0 or more)
- Rule bottom: *conclusion*
- Metasemantics: If all hypotheses hold, then conclusion holds

Rule schemas

$$\frac{e1 \Downarrow \lambda x.\ e3 \quad e2 \Downarrow v2 \quad e3\{v2/x\} = e4 \quad e4 \Downarrow v}{e1\ e2 \Downarrow v}\ [\text{app}]$$

- Each rule is a schema you “instantiate consistently”
- So [app] “works” “for all” x, e1, e2, e3, e4, v2, and v
- But “each” e1 has to be the “same” expression
  - Replace *metavariables* with appropriate terms
  - Deep connection to logic programming (e.g., Prolog)

Instantiating rules

$$\frac{}{\lambda x.\ e \Downarrow \lambda x.\ e}\ [\text{lam}]$$

- Two example legitimate instantiations:
  - $\lambda z.\ z \Downarrow \lambda z.\ z$
    - *x* instantiated with z, *e* instantiated with *z*
  - $\lambda z.\ \lambda y.\ y\ z \Downarrow \lambda z.\ \lambda y.\ y\ z$
    - *x* instantiated with z, *e* instantiated with $\lambda y.\ y\ z$
- Two example illegitimate instantiations:
  - $\lambda z.\ z \Downarrow \lambda y.\ z$
  - $\lambda z.\ \lambda y.\ y\ z \Downarrow \lambda z.\ \lambda z.\ Z$

Must get your rules “just right” so you don’t allow too much or too little

Derivations

- Tuple is “in the relation” if there exists a derivation of it
  – An upside-down (or not?!) tree where each node is an instantiation and leaves are axioms (no hypotheses)
- To show $e \Downarrow v$ for some e and v, *give a derivation*
  – But we rarely “hand-evaluate” like this
  – We’re just defining a semantics remember
- Let’s work through an example derivation for
  $(\lambda x.\ \lambda y.\ y\ x)\ ((\lambda z.\ z)\ (\lambda z.\ z)) \Downarrow \lambda y.\ y\ (\lambda z.\ z)$

Which relation?

So *exactly which* relation did we define
– The pairs at the *bottom of finite-height derivations*

Note: A derivation tree is like the tree of calls in a large-step interpreter
– [when relation is a function]
– Rule being instantiated is branch of the match-expression
– Instantiation is arguments/results of the recursive call

A couple extremes

- This rules are a ***bad idea*** because either one adds all pairs to the relation

$$\frac{}{e \Downarrow v} \qquad\qquad \frac{e1 \Downarrow v1}{e \Downarrow v}$$

- This rule is ***pointless*** because it adds no pairs to the relation

$$\frac{e \Downarrow v}{e \Downarrow v}$$

Summary so far

- Define judgment via a collection of inference rules
  - Tuple in the relation ("judgment holds") if a derivation (tree of instantiations ending in axioms) exists

As an interpreter, could be "nondeterministic":
- Multiple derivations, maybe multiple *v* such that $e \Downarrow v$
  - Our example language is deterministic
  - In fact, "syntax directed" (≤1 rule per syntax form)
- Still need rules for $e\{v/x\}=e'$
- Let's do more judgments (i.e., languages) to get the hang of it…

Call-by-name large-step

$\boxed{e \Downarrow_N v}$ $\boxed{e\{v/x\} = e'}$

$$\frac{}{\lambda x.\ e \Downarrow_N \lambda x.\ e}\ \text{[lam]}$$

$$\frac{e1 \Downarrow_N \lambda x.\ e3 \qquad e3\{e2/x\} = e4 \qquad e4 \Downarrow_N v}{e1\ e2 \Downarrow_N v}\ \text{[app]}$$

- Easier to see the difference than in OCaml
- Formal statement of amazing fact:

  For all *e*, if there exists a *v* such that $e \Downarrow v$ then there exists a *v2* such that $e \Downarrow_N v2$

  (Proof is non-trivial & must reason about substitution)

IMP

- Two judgments $H;e \Downarrow i$ and $H;s \Downarrow H2$
- Assume get(*H*,*x*,*i*) and set(*H*,*x*,*i*,*H2*) are defined
- Let's try writing out inference rules for the judgments…

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Encoding motivation

- Fairly crazy: we left out integers, conditionals, data structures, …
- Turns out we're Turing complete
  - We can encode whatever we need
  - (Just like assembly language can)
- Motivation for encodings
  - Fun and mind-expanding
  - Shows we are not oversimplifying the model ("numbers are syntactic sugar")
  - Can show languages are too expressive Example: C++ template instantiation
- Encodings are also just "(re)definition via translation"

Encoding booleans

The "Boolean Abstract Data Type (ADT)"

- There are 2 booleans and 1 conditional expression
  - The conditional takes 3 (curried) arguments
    - If 1st argument is one bool, return 2nd argument
    - If 1st argument is other bool, return 3rd argument
- Any set of 3 expressions meeting this specification *is* a proper encoding of booleans
- Here is one (of many):
  - "true" `λx. λy. x`
  - "false" `λx. λy. y`
  - "if" `λb. λt. λf. b t f`

Example

- Given our encoding:
  - "true" `λx. λy. x`
  - "false" `λx. λy. y`
  - "if" `λb. λt. λf. b t f`
- We can derive "if" "true" v1 v2 $\Downarrow$ v1
- And every "law of booleans" works out
  - And every non-law does not
- By the way, this is OOP

But…

- Evaluation order matters!
  - With $\Downarrow$, our "if" is not YFL's `if`

"if" "true" (`λx.x`) (λx. x x) (λx. x x) doesn't terminate

but

"if" "true" (`λx.x`) (λz. (λx. x x) (λx. x x) z) terminates

  - Such "thunking" is unnecessary using $\Downarrow_N$

Encoding pairs

- The "Pair ADT"
  - There is 1 constructor and 2 selectors
  - 1st selector returns 1st argument passed to the constructor
  - 2nd selector returns 2nd argument passed to the constructor
- This does the trick:
  - "make_pair" `λx. λy. λz. z x y`
  - "first" `λp. p (λx. λy. x)`
  - "second" `λp. p (λx. λy. y)`
- Example:

  "snd" ("fst" ("make_pair" ("make_pair" v1 v2) v3)) $\Downarrow$ v2

Reusing Lambda

- Is it weird that the encodings of Booleans and pairs both used `(λx. λy. x)` and `(λx. λy. y)` for different purposes?
- Is it weird that the same bit-pattern in binary code can represent an int, a float, an instruction, or a pointer?
- Von Neumann: Bits can represent (all) code and data
- Church (?): Lambdas can represent (all) code and data
- Beware the "Turing tarpit"

Encoding lists

- Why start from scratch? Can build on bools and pairs:
  - "empty-list" is "make_pair" "false" "false"
  - "cons" is `λh.λt.`"make_pair" "true" "make_pair" *h t*
  - "is-empty" is …
  - "head" is …
  - "tail" is …
- Note:
  - Not too far from how lists are implemented
  - Taking "tail" ("tail" "empty") will produce some lambda
    - Just like, without page-protection hardware , `null->tail->tail` would produce some bit-pattern

Encoding natural numbers

- Known as "Church numerals"
  - Will skip in the interest of time
- The "natural number" ADT is basically:
  - "zero"
  - "successor" (the add-one function)
  - "plus"
  - "is-equal"
- Encoding is correct if "is-equal" agrees with elementary-school arithmetic
- [Don't need "full" recursion, but with "full" recursion, can also just do lists of Booleans…]

Recursion

- Can we write *useful* loops? Yes!

To write a recursive function:

- Write a function that takes an `f` and call `f` in place of recursion:
  - Example (in enriched language):
    `λf.λx.if x=0 then 1 else (x * f(x-1))`
- Then apply "fix" to it to get a recursive function
  "fix" `λf.λx.if x=0 then 1 else (x * f(x-1))`
- Details, especially in CBV are icky; but it's possible and need be done only once. *For the curious:*
  "fix" is `λf.(λx.f (λy. x x y))(λx.f (λy. x x y))`

More on "fix"

- "fix" is also known as the Y-combinator
- The informal idea:
  - "`fix`"`(λf.e)` becomes something like
    `e{("fix" (λf.e)) / f}`
  - That's unrolling the recursion once
  - Further unrollings are delayed (happen as necessary)
- Teaser: Most type systems disallow "fix"
  - So later we'll add it as a primitive
  - Example: OCaml can never type-check `(x x)`

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Our goal

Need to define

$\boxed{e1\{e2/x\} = e3}$

- Used in [app] rule
- Informally, "replace occurrences of *x* in *e1* with *e2*"
- Shockingly subtle to get right (in theory or programming)
- (Under call-by-value, only need *e2* to be a value, but that doesn't make it much easier, so define the more general thing.)

Try #1[WRONG]

$\boxed{e1\{e2/x\} = e3}$

$$\frac{}{x\{e/x\} = e} \qquad \frac{y \mathrel{!=} x}{y\{e/x\} = y} \qquad \frac{e1\{e2/x\} = e3}{(\lambda y\,.\,e1)\{e2/x\} = \lambda y\,.\,e3}$$

$$\frac{ea\{e2/x\} = ea' \qquad eb\{e2/x\} = eb'}{(ea\ eb)\ \{e2/x\} = ea'\ eb'}$$

- Recursively replace every x leaf with e2
- But the rule for substituting into (nested) functions is wrong: If the function's argument binds the same variable (shadowing), we should not change the function's body
- Example program: `(λx.λx.x) 42`

Try #2 [WRONG]

$$\boxed{e1\{e2/x\} = e3}$$

$$\frac{}{x\{e/x\} = e} \qquad \frac{y \mathrel{!=} x}{y\{e/x\} = y} \qquad \frac{e1\{e2/x\} = e3 \quad y \mathrel{!=} x}{(\lambda y\,.\,e1)\{e2/x\} = \lambda y\,.\,e3}$$

$$\frac{ea\{e2/x\} = ea' \quad eb\{e2/x\} = eb'}{(ea\ eb)\ \{e2/x\} = ea'\ eb'} \qquad \frac{}{(\lambda x\,.\,e1)\{e2/x\} = \lambda x\,.\,e1}$$

- Recursively replace every x leaf with e2, but respect shadowing
- Still wrong due to *capture [actual technical term]:*
  - *Example:* (λy . e1){y/x}
  - *Example* (λy . e1){(λz.y/x}
  - In general, if "y appears free in e2"

More on capture

- Good news: capture can't happen under CBV or CBN
  - *If* program starts with no unbound ("free") variables
- Bad news: Can still result from "inlining"
- Bad news: It's still "the wrong definition" in general
  - My experience: The nastiest of bugs in language tools

Try #3 [Almost Correct]

- First define an expression's "free variables" (braces here are set notation)
  - FV(x) = {x}
  - FV(e1 e2) = FV(e1) U FV(e2)
  - FV(λy . e) = FV(e) – {y}
- Now require "no capture":

$$\frac{e1\{e2/x\} = e3 \quad y \mathrel{!=} x \quad y \text{ not in } FV(e2)}{(\lambda y\,.\,e1)\{e2/x\} = \lambda y\,.\,e3}$$

Try #3 in Full

$$\boxed{e1\{e2/x\} = e3}$$

$$\frac{}{x\{e/x\} = e} \qquad \frac{y \mathrel{!=} x}{y\{e/x\} = y} \qquad \frac{e1\{e2/x\} = e3 \quad y \mathrel{!=} x \quad y \text{ not in } FV(e2)}{(\lambda y\,.\,e1)\{e2/x\} = \lambda y\,.\,e3}$$

$$\frac{ea\{e2/x\} = ea' \quad eb\{e2/x\} = eb'}{(ea\ eb)\ \{e2/x\} = ea'\ eb'} \qquad \frac{}{(\lambda x\,.\,e1)\{e2/x\} = \lambda x\,.\,e1}$$

- No mistakes with what is here…
- … but only a partial definition
  - What if y *is* in the free-variables of e2

Implicit renaming

$$\frac{e1\{e2/x\} = e3 \quad y \mathrel{!=} x \quad y \text{ not in } FV(e2)}{(\lambda y\,.\,e1)\{e2/x\} = \lambda y\,.\,e3}$$

- But this is a partial definition due to a "syntactic accident", until…
- We allow "implicit, systematic renaming" of any term
  - In general, we never distinguish terms that differ only in variable names
  - A key language-design principle
  - Actual variable choices just as "ignored" as parens
  - Means rule above can "always apply" with a lambda
- Called "alpha-equivalence": terms differing only in names of variables are *the same term*

Try #4 [correct]

- [Includes systematic renaming and drops an unneeded rule]

$$\boxed{e1\{e2/x\} = e3}$$

$$\frac{}{x\{e/x\} = e} \qquad \frac{y \mathrel{!=} x}{y\{e/x\} = y} \qquad \frac{e1\{e2/x\} = e3 \quad y \mathrel{!=} x \quad y \text{ not in } FV(e2)}{(\lambda y\,.\,e1)\{e2/x\} = \lambda y\,.\,e3}$$

$$\frac{ea\{e2/x\} = ea' \quad eb\{e2/x\} = eb'}{(ea\ eb)\ \{e2/x\} = ea'\ eb'} \qquad \cancel{\frac{}{(\lambda x\,.\,e1)\{e2/x\} = \lambda x\,.\,e1}}$$

More explicit approach

- While "everyone in the PL field":
  - Understands the capture problem
  - Avoids it by saying "implicit systematic renaming"

  you may find that unsatisfying…

  … especially if you have to implement substitution while avoiding capture

- So this more explicit version also works ("fresh z for y"):

$$\frac{z \text{ not in } FV(e1) \cup FV(e2) \cup \{x\} \quad e1\{z/y\} = e3 \quad e3\{e2/x\} = e4}{(\lambda y.\, e1)\{e2/x\} = \lambda z.\, e4}$$

  - You have to "find an appropriate *z*", but one always exists and `__$$tmp` appended to a global counter "probably works"

Note on metasyntax

- Substitution often thought of as a metafunction, not a judgment
  - I've seen many nondeterministic languages
  - But never a nondeterministic definition of substitution
- So instead of writing:

$$\frac{e1 \Downarrow \lambda x.\, e3 \quad e2 \Downarrow v2 \quad e3\{v2/x\} = e4 \quad e4 \Downarrow v}{e1\ e2 \Downarrow v}\ [\text{app}]$$

- Just write:

$$\frac{e1 \Downarrow \lambda x.\, e3 \quad e2 \Downarrow v2 \quad e3\{v2/x\} \Downarrow v}{e1\ e2 \Downarrow v}\ [\text{app}]$$

What will we do

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  - Notes on concrete syntax (relates to OCaml)
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- Define substitution precisely
- Environments

Next time??

- Small-step
- Play with *continuations* ("very fancy" language feature)

Where we're going

- Done: large-step for untyped lambda-calculus
  - CBV and CBN
  - Note: infinite number of other "reduction strategies"
  - Amazing fact: all equivalent if you ignore termination!

- Now other semantics, all equivalent to CBV:
  - With environments (in OCaml to prep for Homework 3)
  - Basic small-step (easy)
  - Contextual semantics (similar to small-step)
    - Leads to precise definition of *continuations*

Slide repeat…

```
type exp = Var of string
         | Lam of string*exp
         | Apply of exp * exp
exception BadExp
let subst e1_with e2_for x = …(*to be discussed*)
let rec interp_large e =
  match e with
   Var _ -> raise BadExp(*unbound variable*)
  | Lam _ -> e (*functions are values*)
  | Apply(e1,e2) ->
     let v1 = interp_large e1 in
     let v2 = interp_large e2 in
     match v1 with
       Lam(x,e3) -> interp_large (subst e3 v2 x)
     | _ -> failwith "impossible" (* why? *)
```


Environments

- Rather than substitute, let's keep a map from variables to values
  - Called an environment
  - Like IMP's heap, but immutable and 1 not enough
- So a program "state" is now exp and environment
- A function body is evaluated under the environment where it was defined!
  - Use closures to store the environment
  - See also Lecture 1

No more substitution

```
type exp = Var of string
         | Lam of string * exp
         | Apply of exp * exp
         | Closure of string * exp * env
and env = (string * exp) list

let rec interp env e =
  match e with
   Var s -> List.assoc s env (* do the lookup *)
 | Lam(s,e2) -> Closure(s,e2,env) (* store env! *)
 | Closure _ -> e (* closures are values *)
 | Apply(e1,e2) ->
    let v1 = interp env e1 in
    let v2 = interp env e2 in
    match v1 with
      Closure(s,e3,env2) -> interp((s,v2)::env2) e3
     | _ -> failwith "impossible"
```


Worth repeating

- A closure is a pair of code and environment
  - Implementing higher-order functions is not magic or run-time code generation
- An okay way to think about OCaml
  - Like thinking about OOP in terms of vtables
- Need not store whole environment of course
  - See Homework 3

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  - Notes on concrete syntax (relates to OCaml)
  - Define semantics with inference rules
- Lambda encodings (show our language is mighty)
- Define substitution precisely
  - And revisit function equivalences
- Environments

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